TWK2A
The Inverse Laplace transform (Section 7.2)
Solutions

1.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-2\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\}\\ &= t-2t^4\end{aligned}$$

2.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\left(\frac{2}{s}-\frac{1}{s^3}\right)^2\right\} &= \mathcal{L}^{-1}\left\{\frac{4}{s^2}-\frac{4}{s^4}+\frac{1}{s^6}\right\}\\ &= 4t-4\frac{t^3}{3!}+\frac{t^5}{5!}\\ &= 4t-2\frac{t^3}{3}+\frac{t^5}{120}\end{aligned}$$

3.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{4s+1}\right\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{4}}\right\}\\ &= \frac{1}{4}e^{-\frac{1}{4}t}\end{aligned}$$

4.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} &= 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}\\ &= 10\cos 4t\end{aligned}$$

5.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-2\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\\ &= 2\cos 3t-2\sin 3t\end{aligned}$$

6.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+5)(s-4)}\right\}\\ &= -\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}+\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}\\ &= -\frac{1}{9}e^{-5t}+\frac{1}{9}e^{4t}\end{aligned}$$

7.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{s^2+4s+4}{s^3}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{4}{s^2}+\frac{4}{s^3}\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+4\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+2\,\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}\\ &= 1+4t+2t^2\end{aligned}$$

8.

$$f(t) \equiv \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\}$$

Let

$$\frac{s+1}{s(s-4)} = \frac{A}{s}+\frac{B}{s-4} = \frac{A(s-4)+Bs}{s(s-4)}$$

Equate the numerators:

$$\begin{aligned} & A(s-4)+Bs &= s+1 & \\ \text{Put } s=0: \quad & -4A &= 1; & \quad A=-\frac{1}{4}\\ \text{Put } s=4: \quad & 4B &= 5; & \quad B=\frac{5}{4}\end{aligned}$$

We now have

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s}+\frac{\frac{5}{4}}{s-4}\right\}\\ &= -\frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{5}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}\\ &= -\frac{1}{4}+\frac{5}{4}\,e^{4t}.\end{aligned}$$

9.

$$f(t) \equiv \mathcal{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{6s+3}{(s^2+1)(s^2+4)}\right\}$$

Let

$$\frac{6s+3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Equating the numerators, we have:

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) = 6s+3$$

Compare the coefficients of s :

$$\left.\begin{aligned} s^3: &\quad A+C=0 \\ s: &\quad 4A+C=6 \end{aligned}\right\} \quad \Rightarrow \quad A=2, \quad C=-2$$

$$\left.\begin{aligned} s^2: &\quad B+D=0 \\ s^0\,(=1): &\quad 4B+D=3 \end{aligned}\right\} \quad \Rightarrow \quad B=1, \quad D=-1$$

Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1} - \frac{2s+1}{s^2+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} + \frac{1}{s^2+1} - \frac{2s}{s^2+4} - \frac{1}{s^2+4}\right\} \\ &= 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= 2\cos t + \sin t - 2\cos 2t - \frac{1}{2}\sin 2t. \end{aligned}$$

10. Assume

$$\frac{s^2+1}{s\underbrace{(s-1)\,(s+1)}_{s^2-1}\,(s-2)} = \frac{A}{s} + \frac{Bs+C}{s^2-1} + \frac{D}{s-2}$$

$$\Rightarrow \left\{\begin{aligned} A+B+D &= 0 \\ -2A-2B+C &= 1 \\ -A-2C-D &= 0 \\ 2A &= 1 \end{aligned}\right.$$

$$\Rightarrow \left\{\begin{aligned} &A = \tfrac{1}{2} \\ &B = -\tfrac{4}{3} \\ &C = -\tfrac{2}{3} \\ &D = \tfrac{5}{6} \end{aligned}\right.$$

$$\Rightarrow \frac{s^2+1}{s\left(s-1\right)\left(s+1\right)\left(s-2\right)} = \frac{1}{2}\left(\frac{1}{s}\right)-\frac{4}{3}\left(\frac{s}{s^2-1}\right)-\frac{2}{3}\left(\frac{1}{s^2-1}\right)+\frac{5}{6}\left(\frac{1}{s-2}\right)$$

Now,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)\right\} &= 1 \\
\mathcal{L}^{-1}\left\{\left(\frac{s}{s^2-1}\right)\right\} &= \cosh t = \frac{e^{kt}+e^{-kt}}{2} \\
\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2-1}\right)\right\} &= \sinh t = \frac{e^{kt}-e^{-kt}}{2} \\
\mathcal{L}^{-1}\left\{\left(\frac{1}{s-2}\right)\right\} &= e^{2t}
\end{aligned}$$

and so

$$\mathcal{L}^{-1}\left\{\frac{s^2+1}{s\left(s-1\right)\left(s+1\right)\left(s-2\right)}\right\} = \frac{1}{2} - e^{t} - \frac{e^{-t}}{3} + \frac{5e^{2t}}{6}.$$

11. We have

$$\begin{aligned}
\frac{\left(s+1\right)^3}{s^4} &= \frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}+\frac{1}{s^4} \\
&= \frac{1}{s}+3\left(\frac{1}{s^2}\right)+\frac{3}{2}\left(\frac{2}{s^3}\right)+\frac{1}{6}\left(\frac{6}{s^4}\right)
\end{aligned}$$

and so

$$\mathcal{L}^{-1}\left\{\frac{\left(s+1\right)^3}{s^4}\right\} = 1+3t+\frac{3t^2}{2}+\frac{t^3}{6}.$$

12.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
&= 1-e^{-t}
\end{aligned}$$

where we used

$$
\begin{aligned}
\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow &\quad A(s+1) + Bs = 1 \\
\Rightarrow &\quad A + B = 0 \ \text{ and } \ A = 1 \\
\Rightarrow &\quad B = -1
\end{aligned}
$$