TWK2A
The Laplace transform (Section 7.1)
Solutions

1.

$$
\begin{aligned}
f(t) &= \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases} \\
\mathcal{L}\{f\} &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^1 e^{-st} t dt + \int_1^\infty e^{-st} dt \\
&= \left[\frac{-e^{-st}t}{s}\right]_0^1 + \frac{1}{s}\int_0^1 e^{-st} dt + \int_1^\infty e^{-st} dt \\
&= -\frac{e^s}{s} + \frac{1}{s}\left[\frac{-e^{-st}}{s}\right]_0^1 + \left[\frac{-e^{-st}}{s}\right]_1^0 \\
&= \frac{-e^{-s}}{s} + \frac{1}{s}\left[\frac{-e^{-s}}{s} + \frac{1}{s}\right] + \frac{e^{-s}}{s} \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s^2} \qquad (s > 0)
\end{aligned}
$$

2.

$$
\begin{aligned}
f(t) &= \left\{ \begin{array}{cc} 2t+1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{array} \right] \\
\mathcal{L}\{f\} &= \int_0^\infty e^{-st} f(t) dt \\
&= \int_0^1 e^{-st}(2t+1)dt + \int_1^\infty e^{-st} 0 dt \\
&= 2\int_0^1 e^{-st} t dt + \int_0^1 e^{-st} dt \\
&= 2\left(\left[\frac{-e^{-st}t}{s} \right]_0^1 + \frac{1}{s}\int_0^1 e^{-st} dt \right) + \int_0^1 e^{-st} dt \\
&= 2\left(\frac{-e^{-s}}{s} + \frac{1}{s}\left[\frac{-e^{-st}}{s} \right]_0^1 \right) + \left[\frac{-e^{-st}}{s} \right]_0^1 \\
&= \frac{-2e^{-s}}{s} - \frac{2e^{-s}}{s^2} + \frac{2}{s^2} - \frac{e^{-s}}{s} + \frac{1}{s} \\
&= \frac{-3e^{-s}}{s} - \frac{2e^{-s}}{s^2} + \frac{1}{s} + \frac{2}{s^2} \qquad (s > 0)
\end{aligned}
$$

3.

$$
\begin{aligned}
f(t) &= e^{-2t-5} \\
\mathcal{L}\{f\} &= \int_0^{\infty} e^{-st} e^{-2t-5} dt \\
&= e^{-5} \int\limits_0^{\infty} e^{-(s+2)t} dt \\
&= e^{-5} \left[\frac{-e^{(s+2)t}}{(s+2)} \right]_0^{\infty} \\
&= e^{-5} \left(\frac{1}{s+2} \right) \\
&= \frac{e^{-5}}{s+2} \qquad (s > -2)
\end{aligned}
$$

4.

$$
\begin{aligned}
f(t) &= \begin{cases} 0, & 0 \leqslant t < a \\ c, & a \leqslant t < b \\ 0, & t \geqslant b \end{cases} \\
\mathcal{L}\{f(t)\} &= 0 + \int_a^b ce^{-st}\, dt + 0 \\
&= \frac{c}{(-s)} e^{-st} \Big|_a^b \\
&= \frac{c(e^{-sa} - e^{-sb})}{s} \quad ; \qquad s > 0
\end{aligned}
$$

5.

$$
\begin{aligned}
\mathcal{L}\{t^2e^{-2t}\} &= \int_0^\infty t^2 e^{-2t} e^{-st}\, dt \\
&= \int_0^\infty t^2 e^{-(s+2)t}\, dt \\
&= t^2 \frac{e^{-(s+2)t}}{-(s+2)}\bigg|_0^\infty - \int_0^\infty 2t \frac{e^{-(s+2)t}}{-(s+2)}\, dt \\
&= 0 - 2\left[t \frac{e^{-(s+2)t}}{(s+2)^2}\bigg|_0^\infty - \int_0^\infty \frac{e^{-(s+2)t}}{(s+2)^2}\, dt \right] \qquad (s > -2) \\
&= 0 - 2\ \frac{e^{-(s+2)t}}{(s+2)^3}\bigg|_0^\infty \\
&= \frac{2}{(s+2)^3}
\end{aligned}
$$

6.

$$
\begin{aligned}
F(s) &= \int_0^2 4e^{-st} dt \\
&= -\frac{4}{s}\ e^{-st}\Big|_0^2 \\
&= -\frac{4}{s}\left(e^{-2s} - 1\right) \\
&= \frac{4\left(1 - e^{-2s}\right)}{s}, \quad s > 0
\end{aligned}
$$

7.

$$
\begin{aligned}
F(s) &= \int_0^{\pi} e^{-st}\sin t\,dt \\
&= \left(-\frac{1}{s}\right)e^{-st}\sin t\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{s}\right)e^{-st}\cos t\,dt \\
&= 0 + \frac{1}{s}\int_0^{\pi} e^{-st}\cos t\,dt \\
&= \left(-\frac{1}{s^2}\right)e^{-st}\cos t\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{s^2}\right)e^{-st}\,(-\sin t)\,dt \\
&= \left(-\frac{1}{s^2}\right)\left(-e^{-\pi s} - 1\right) - \frac{1}{s^2}\int_0^{\pi} e^{-st}\sin t\,dt \\
&= \frac{1+e^{-\pi s}}{s^2} - \frac{1}{s^2}F(s) \\
\Rightarrow F(s) &= \frac{1+e^{-\pi s}}{1+s^2}.
\end{aligned}
$$

8.

$$
\begin{aligned}
F(s) &= \int_0^{\infty} e^{-st}e^{t}\cos t\,dt \\
&= \int_0^{\infty} e^{(1-s)t}\cos t\,dt \\
&= \left(\frac{1}{1-s}\right)e^{(1-s)t}\cos t\Big|_0^{\infty} - \int_0^{\infty}\left(\frac{1}{1-s}\right)e^{(1-s)t}(-\sin t)\,dt \\
&= \frac{0-1}{1-s} + \frac{1}{1-s}\int_0^{\infty} e^{(1-s)t}\sin t\,dt \qquad (s>1) \\
&= \frac{1}{s-1} + \frac{1}{1-s}\left[\left(\frac{1}{1-s}\right)e^{(1-s)t}\sin t\Big|_0^{\infty} - \int_0^{\infty}\left(\frac{1}{1-s}\right)e^{(1-s)t}\cos t\,dt\right] \\
&= \frac{1}{s-1} + 0 - \frac{1}{(s-1)^2}\int_0^{\infty} e^{(1-s)t}\cos t\,dt \\
&= \frac{1}{s-1} - \frac{1}{(s-1)^2}F(s)
\end{aligned}
$$

Make $F(s)$ the subject of the equation:

$$\begin{aligned}\left[(s-1)^2+1\right]F(s) &= s-1 \\ F(s) &= \frac{s-1}{(s-1)^2+1} = \frac{s-1}{s^2-2s+2}, \qquad s>1\end{aligned}$$

9.

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st}t\cos t\,dt \\ &= \left(-\frac{1}{s}\right)e^{-st}t\cos t\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}\right)e^{-st}(\cos t - t\sin t)\,dt \\ &= 0+\frac{1}{s}\int_0^\infty e^{-st}\cos t\,dt - \frac{1}{s}\int_0^\infty e^{-st}t\sin t\,dt \qquad (s>0) \\ &= \frac{1}{s}\left(\frac{s}{s^2+1}\right) - \frac{1}{s}\left[\left(-\frac{1}{s}\right)e^{-st}t\sin t\Big|_0^\infty - \int_0^\infty\left(-\frac{1}{s}\right)e^{-st}(\sin t + t\cos t)\,dt\right] \\ &= \frac{1}{s^2+1} - 0 - \frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt - \frac{1}{s^2}\int_0^\infty t\cos t\,dt \\ &= \frac{1}{s^2+1} - \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) - \frac{1}{s^2}F(s)\end{aligned}$$

Make $F(s)$ the subject of the equation:

$$\begin{aligned}(s^2+1)F(s) &= \frac{s^2}{s^1+1} - \frac{1}{s^2+1} = \frac{s^2-1}{s^2+1} \\ F(s) &= \frac{s^2-1}{(s^2+1)^2}, \qquad s>0\end{aligned}$$

10.

$$\cosh kt = \frac{e^{kt}+e^{-kt}}{2} \Rightarrow \mathcal{L}\{\cosh kt\} = \frac{1}{2}\left(\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right).$$

Now,

$$\begin{aligned}\mathcal{L}\{e^{kt}\} &= \int_0^\infty e^{-st}e^{kt}dt = \int_0^\infty e^{(-s+k)t}dt \\ &= \left[\frac{e^{(-s+k)t}}{-s+k}\right]_0^\infty = \frac{1}{s-k}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\left\{e^{-kt}\right\} &= \int\limits_0^\infty e^{-st}e^{-kt}dt = \int\limits_0^\infty e^{(-s-k)t}dt \\ &= \left[\frac{e^{(-s-k)t}}{-s-k}\right]_0^\infty = \frac{1}{s+k}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}\left\{\cosh kt\right\} &= \frac{1}{2}\left(\frac{1}{s-k}+\frac{1}{s+k}\right) = \frac{1}{2}\left(\frac{2s}{(s-k)(s+k)}\right) \\ &= \frac{s}{s^2-k^2}.\end{aligned}$$

11.

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\} &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^{\frac{\pi}{2}} 0e^{-st}dt + \int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\,dt = \int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\,dt\end{aligned}$$

Now,

$$\begin{aligned}\int_{\frac{\pi}{2}}^\infty e^{-st}\cos tdt &= \left[e^{-st}\sin t\right]_{\frac{\pi}{2}}^\infty - \int_{\frac{\pi}{2}}^\infty (-se^{-st})\sin tdt \\ &= -e^{-s\frac{\pi}{2}} + s\int_{\frac{\pi}{2}}^\infty e^{-st}\sin tdt \qquad\qquad s>0 \\ &= -e^{-s\frac{\pi}{2}} + s\left(\left[-e^{-st}\cos t\right]_{\frac{\pi}{2}}^\infty + \int_{\frac{\pi}{2}}^\infty (-se^{-st})\cos tdt\right) \\ &= -e^{-s\frac{\pi}{2}} - s^2\int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\,dt\end{aligned}$$

and so

$$\mathcal{L}\left\{f(t)\right\} = \int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\,dt = \frac{-e^{-s\frac{\pi}{2}}}{1+s^2}.$$

12.

$$
\begin{aligned}
f(t) &= t^2 + 6t - 3 \\
\mathcal{L}\{f\} &= \mathcal{L}\{t^2 + 6t - 3\} \\
&= \mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} - 3\mathcal{L}\{1\} \\
&= \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}.
\end{aligned}
$$

13.

$$
\begin{aligned}
f(t) &= (2t-1)^3 = (2t-1)(4t^2 - 4t + 1) = 8t^3 - 12t^2 + 6t - 1 \\
\mathcal{L}\{f\} &= 8\mathcal{L}\{t^3\} - 12\mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} - \mathcal{L}\{1\} \\
&= \frac{8 \cdot 3!}{s^4} - \frac{12 \cdot 2!}{s^3} + \frac{6}{s^2} - \frac{1}{s} \\
&= \frac{48}{s^4} - \frac{24}{s^3} + \frac{6}{s^2} - \frac{1}{s}.
\end{aligned}
$$

14.

$$
\begin{aligned}
f(t) &= \cos 5t + \sin 2t \\
\mathcal{L}\{f\} &= \mathcal{L}\{\cos 5t\} + \mathcal{L}\{\sin 2t\} \\
&= \frac{s}{s^2 + 25} + \frac{2}{s^2 + 4}.
\end{aligned}
$$