TWK2A
Nonlinear DEs (Section 4.9)
Solutions

1.

$$y_1' = y_1'' = 0, \qquad y_1 y_1'' = 0 = \frac{1}{2}(y_1')^2.$$

$$y_2' = 2x, \quad y_2'' = 2, \qquad y_2 y_2'' = 2x^2 = \frac{1}{2}(2x)^2 = \frac{1}{2}(y_2')^2$$

$$\begin{aligned} y \equiv c_1 y_1 + c_2 y_2 &= c_1 + c_2 x^2 \\ \Rightarrow y' &= 2c_2 x \\ \Rightarrow y'' &= 2c_2 \\ \Rightarrow yy'' &= (c_1 + c_2 x^2) 2c_2 = 2c_1 c_2 + 2c_2^2 x^2 \\ \Rightarrow \frac{1}{2}(y')^2 &= 2c_2^2 x^2. \end{aligned}$$

To satisfy the equation $yy'' = \frac{1}{2}(y')^2$ we must have $c_1 c_2 = 0$. In other words $c_1 = 0$ or $c_2 = 0$, and y is one of y_1 and y_2 up to the constant. Thus an arbitrary linear combination of y_1 and y_2 is not a solution to the equation whenever $c_1 \neq 0$ and $c_2 \neq 0$.

2. Using $u = y'$ and assuming $y = y(x)$

$$
\begin{aligned}
y'' = \frac{du}{dx} &= \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy} \\
\Rightarrow (y+1)u\frac{du}{dy} &= u^2 \\
\Rightarrow \frac{1}{u}\frac{du}{dy} &= \frac{1}{y+1}, \qquad u \neq 0, y \neq -1 \\
\Rightarrow \int \frac{du}{u} &= \int \frac{dy}{y+1} \\
\Rightarrow \ln|u| &= \ln|y+1| + c \\
\Rightarrow u &= c_1(y+1), \qquad c_1 \equiv e^c \\
\Rightarrow \frac{dy}{dx} &= c_1(y+1) \\
\Rightarrow \int \frac{dy}{y+1} &= \int c_1 dx \\
\Rightarrow \ln|y+1| &= c_1 x + k \\
\Rightarrow y &= c_2 e^{c_1 x} - 1, \qquad c_2 \equiv e^k
\end{aligned}
$$

Obviously $y = -1$ is also a solution which can be obtained by setting $c_2 = 0$. After the substitution, $u = 0$ is also easily verified as a solution. Thus $y = C$, where C is a constant, is also solution which can be obtained by setting $c_1 = 0$ in our final solution.

3. Using $u = y'$ and assuming $y = y(x)$

$$
\begin{aligned}
y'' = \frac{du}{dx} &= \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy} \\
\Rightarrow u\frac{du}{dy} + 2yu^3 &= 0 \\
\Rightarrow \int \frac{du}{u^2} &= -2\int y dy, \qquad u \neq 0 \\
\Rightarrow -\frac{1}{u} &= -y^2 + c_1 \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{y^2 - c_1} \\
\Rightarrow \int (y^2 - c_1) dy &= \int dx \\
\Rightarrow \frac{1}{3}y^3 - c_1 y &= x + c_2.
\end{aligned}
$$

From $u = 0$ we also obtain the constant solution $y = C$. We can always test our solution to verify that it satisfies the differential equation. We differentiate our solution twice

$$
\begin{aligned}
\frac{d}{dx} y^2 y' - c_1 y' &= 1 \\
\Rightarrow \frac{d^2}{dx^2} 2y(y')^2 + y^2 y'' - c_1 y'' &= 0 \\
(\times y') \Rightarrow 2y(y')^3 + y^2 y' y'' - c_1 y' y'' &= 2y(y')^3 + (y^2 y' - c_1 y') y'' \\
\Rightarrow 2y(y')^3 + y'' &= 0.
\end{aligned}
$$

4. Substituting $u = y'$ yields

$$\begin{aligned}
y'' &= \frac{du}{dx} \\
\Rightarrow x\frac{du}{dx} &= u + u^3 \\
\Rightarrow x\frac{1}{u^3}\frac{du}{dx} - \frac{1}{u^2} &= 1 \\
\Rightarrow w \equiv \frac{1}{u^2}, \qquad \frac{dw}{dx} &= -2\frac{1}{u^3}\frac{du}{dx} \\
\Rightarrow -\frac{x}{2}\frac{dw}{dx} - w &= 1 \\
\Rightarrow \frac{dw}{dx} + \frac{2}{x}w &= -\frac{2}{x}, \qquad x \neq 0 \\
\Rightarrow \text{integrating factor } e^{\int \frac{2}{x}dx} &= x^2 \\
\Rightarrow \frac{d}{dx}\left(x^2 w\right) &= -2x \\
\Rightarrow x^2 w &= -x^2 + c_1 \\
\Rightarrow \frac{1}{u^2} &= \frac{c_1}{x^2} - 1 \\
\Rightarrow u = \frac{dy}{dx} &= \pm\sqrt{\frac{x^2}{c_1 - x^2}} \\
\Rightarrow y(x) &= \pm\sqrt{c_1 - x^2} + c_2
\end{aligned}$$

5. We use the substitution $u = y''$

$$\begin{aligned}
u' &= \sqrt{1+u^2} \\
\Rightarrow \int \frac{du}{\sqrt{1+u^2}} &= \int dx \\
u \equiv \tan t, \qquad du &= \sec^2 t\, dt \\
\Rightarrow \int \frac{\sec^2 t}{\sec t} dt &= x + c \\
\Rightarrow \ln|\sec t + \tan t| &= x + c \\
\Rightarrow \sec t + \tan t &= c_1 e^x, \qquad c_1 \equiv e^c \\
\Rightarrow \sqrt{1+u^2} + u &= c_1 e^x \\
\Rightarrow 1 + u^2 &= c_1^2 e^{2x} - 2c_1 u e^x + u^2 \\
\Rightarrow 2c_1 u &= c_1^2 e^x - e^{-x} \\
\Rightarrow 2c_1 y'' &= c_1^2 e^x - e^{-x} \\
\Rightarrow 2c_1 y' &= c_1^2 e^x + e^{-x} + c_2 \\
\Rightarrow 2c_1 y &= c_1^2 e^x - e^{-x} + c_2 x + c_3 \\
\Rightarrow y(x) &= \frac{1}{2c_1}\left(c_1^2 e^x - e^{-x} + c_2 x + c_3\right).
\end{aligned}$$

6. Note that the dependent variable y is missing. We use the substitution

$$u = y' (\Rightarrow u' = y'')$$

to obtain

$$\begin{aligned}
u' + u^2 + 1 &= 0 \\
\Rightarrow \frac{du}{u^2+1} &= -dx \\
\Rightarrow \arctan u &= -x + c \\
\Rightarrow u &= \tan(-x + c) \\
\Rightarrow y' &= \tan(-x + c) \\
\Rightarrow y &= \ln|\cos(x - c)|.
\end{aligned}$$