TWK2A
Systems of Linear DEs (Section 4.8)
Solutions

1.

$$Dx = -y + t \qquad Dy = x - t$$

$$Dx + y - t = 0 \qquad Dy - x + t = 0$$

$$\begin{aligned} &\Rightarrow \; D^2x + Dy - 1 = 0 \\ &\Rightarrow \; D^2x + (x - t) - 1 = 0 \\ &\Rightarrow \; D^2x + x = t + 1 \end{aligned}$$

Solve $D^2x + x - 1 = 0$ to obtain $x_c(t) = c_1 \cos(t) + c_2 \sin(t)$
Particular solution: Assume $x_p(t) = At + B$
So

$$\begin{aligned} D^2x + x &= At + B = t + 1 \\ \Rightarrow A &= 1, \quad B = 1 \end{aligned}$$

So $x(t) = x_c + x_p = c_1 \cos(t) + c_2 \sin(t) + t + 1$
Now,

$$\begin{aligned} y &= t - Dx \\ &= t - D(c_1 \cos(t) + c_2 \sin(t) + t + 1) \\ &= t + c_1 \sin(t) - c_2 \cos(t) - 1 \end{aligned}$$

So

$$y(t) = c_1 \sin(t) - c_2 \cos(t) + t - 1$$

2.

$$
\begin{aligned}
& (D^2+5)x - 2y = 0; \qquad -2x + (D^2+2)y = 0 \\
\Rightarrow\ & (D^2+2)(D^2+5)x - (D^2+2)2y = 0; \qquad -4x + (D^2+2)2y = 0 \\
\Rightarrow\ & (D^4+7D^2+10)x - (D^2+2)2y = 0; \qquad -4x + (D^2+2)2y = 0 \\
\Rightarrow\ & (D^4+7D^2+10)x - 4x = 0 \\
\Rightarrow\ & D^4x + 7D^2x + 6x = 0 \\
\Rightarrow\ & m^4 + 7m^2 + 6 = 0 \\
\Rightarrow\ & m_1, m_2 = \pm\sqrt{6}i; \quad m_3, m_4 = \pm i \\
& \text{(repeated complex roots)}
\end{aligned}
$$

So $x(t) = c_1 \cos\sqrt{6}t + c_2 \sin\sqrt{6}t + c_3 \cos t + c_4 \sin t$

Now

$$
\begin{aligned}
y &= \frac{1}{2}\left(D^2+5\right)x \\
&= \frac{1}{2}\left(D^2x + 5x\right) \\
&= \frac{1}{2}\left(D^2\left(c_1\cos(\sqrt{6}t) + c_2\sin(\sqrt{6}t) + c_3\cos t + c_4 \sin t\right)\right. \\
&\qquad \left. +5\left(C_1\cos\sqrt{6}t + c_2\sin\sqrt{6}t + c_3\cos t + c_4\sin t\right)\right) \\
&= \frac{1}{2}\left(-6c_1\cos(\sqrt{6}t) - 6c_2\sin\sqrt{6}t - c_3\cos t - c_4\sin t\right. \\
&\qquad \left. + 5c_1\cos\sqrt{6}t + 5c_2\sin\sqrt{6}t + 5c_3\cos t + 5c_4\sin t\right) \\
\Rightarrow y(t) &= -\frac{1}{2}c_1\cos\sqrt{6}t - \frac{1}{2}c_2\sin\sqrt{6}t + 2c_3\cos t + 2c_4\sin t
\end{aligned}
$$

3.

$$\begin{aligned}
& D^2x + Dy = -5x \qquad Dx + Dy = -x + 4y \\
\Rightarrow\ & (D^2+5)x + Dy = 0; \qquad (D+1)x + (D-4)y = 0 \\
\Rightarrow\ & (D-4)(D^2+5)x + (D-4)Dy = 0; \qquad D(D+1)x + D(D-4)y = 0 \\
\Rightarrow\ & (D-4)(D^2+5)x - D(D+1)x = 0 \\
\Rightarrow\ & (D^3 - 4D^2 + 5D - 20)x - (D^2 + D)x = 0 \\
\Rightarrow\ & (D^3 - 5D^2 + 4D - 20)x = 0 \\
\Rightarrow\ & m^3 - 5m^2 + 4m - 20 = 0 \\
\Rightarrow\ & (m-5)(m^2+4) = 0 \\
\Rightarrow\ & m_1 = 5, \quad m_2 = 2i, \quad m_3 = -2i
\end{aligned}$$

So

$$x(t) = c_1e^{5t} + c_2\cos(2t) + c_3\sin(2t)$$

For $y(t)$ we have

$$Dy = -5x - D^2x$$

So

$$\begin{aligned}
Dx - 5x - D^2x &= -x + 4y \\
\Rightarrow y &= \frac{1}{4}\left(Dx - 4x - D^2x\right) \\
&= \frac{1}{4}\left(5c_1e^{5t} - 2c_2\sin(2t) + 2c_3\cos(2t)\right. \\
&\quad -4c_1e^{5t} - 4c_2\cos(2t) - 4c_3\sin(2t) \\
&\quad \left. -\left(25c_1e^{5t} - 4c_2\cos(2t) - 4c_3\sin(2t)\right)\right) \\
&= \frac{1}{4}\left(-24c_1e^{5t} - 2c_2\sin(2t) + 2c_3\cos(2t)\right) \\
\Rightarrow y(t) &= -6c_1e^{5t} - \frac{1}{2}c_2\sin(2t) + \frac{1}{2}c_3\cos(2t)
\end{aligned}$$

4.

$$\begin{aligned}
Dx + z &= e^t \\
(D-1)x + Dy + Dz &= 0 \\
x + 2y + Dz &= e^t \ \left(\Rightarrow Dz = e^t - x - 2y\right)
\end{aligned}$$

So

$$
\begin{aligned}
D^2 + Dz &= e^t; \quad (D-1)x + Dy + Dz = 0 \\
&\Rightarrow D^2x + e^t - x - 2y = e^t; \qquad (D-1)x + Dy + e^t - x - 2y = 0 \\
&\Rightarrow (D^2-1)x - 2y = 0; \qquad (D-2)x + (D-2)y = -e^t \\
&\Rightarrow (D-2)(D^2-1)x - (D-2)2y = 0; \qquad (D-2)2x + (D-2)2y = -2e^t \\
&\Rightarrow (D-2)(D^2-1)x + (D-2)2x = -2e^t \\
&\Rightarrow (D^3 - 2D^2 - D + 2)x + 2Dx - 4x = -2e^t \\
&\Rightarrow D^3x - 2D^2x + Dx - 2x = -2e^t \qquad (1) \\
&\Rightarrow m^3 - 2m^2 + m - 2 = 0 \\
&\Rightarrow m_1 = 2, \quad m_2 = \mathbf{i}, \quad m_3 = -\mathbf{i}.
\end{aligned}
$$

So

$$x_c(t) = c_1e^{2t} + c_2\cos(t) + c_3\sin(t)$$

For particular solution assume $x_p(t) = Ae^t$. So, from (1),

$$
\begin{aligned}
Ae^t - 2Ae^t + Ae^t - 2Ae^t &= -2e^t \\
\Rightarrow A - 2A + A - 2A &= 2A = -2 \\
\Rightarrow A &= -1.
\end{aligned}
$$

Hence,

$$x(t) = c_1e^{2t} + c_2\cos(t) + c_3\sin(t) - e^t.$$

Now

$$
\begin{aligned}
y &= \frac{1}{2}\left(D^2 - 1\right)x = \frac{1}{2}\left(D^2x - x\right) \\
&= \frac{1}{2}\left(4c_1e^{2t} - c_2\cos(t) - c_3\sin(t) - e^t\right. \\
&\quad \left. - c_1e^{2t} - c_2\cos(t) - c_3\sin(t) + e^t\right) \\
&= \frac{3}{2}c_1e^{2t} - c_2\cos(t) - c_3\sin(t).
\end{aligned}
$$

and

$$
\begin{aligned}
z &= e^t - Dx \\
&= e^t - 2c_1e^{2t} - c_2\sin(t) + c_3\cos(t) - e^t \\
&= -2c_1e^{2t} - c_2\sin(t) + c_3\cos(t).
\end{aligned}
$$

5.

$$
\begin{aligned}
& Dx = y - 1 \qquad Dy = -3x + 2y \qquad x(0) = 0 \quad y(0) = 0 \\
\Rightarrow\ & -3Dx = -3y + 3; \qquad D^2y = -3Dx + 2Dy \\
\Rightarrow\ & D^2y = -3y + 3 + 2Dy \\
\Rightarrow\ & D^2y - 2Dy + 3y = 3 \\
\Rightarrow\ & m^2 - 2m + 3 = 0 \\
\rightarrow\ & m_1 = 1, \quad m_2 = 1 \ \text{(repeated real roots)}
\end{aligned}
$$

So

$$y_c(t) = c_1 e^t + c_2 t e^t.$$

For particular solution assume $y_p(t) = A$. Thus,

$$
\begin{aligned}
0 - 2 + 3A &= 3 \\
\Rightarrow A &= \frac{5}{3}
\end{aligned}
$$

and so

$$y(t) = c_1 e^t + c_2 t e^t + \frac{5}{3}.$$

Now,

$$
\begin{aligned}
x &= \frac{1}{3}(2y - Dy) \\
&= \frac{1}{3}\left(2c_1 e^t + 2c_2 t e^t + \frac{10}{3} - c_1 e^t - c_2 t e^t - c_2 e^t\right) \\
&= \frac{1}{3}\left((c_1 - c_2)e^t + c_2 t e^t + \frac{10}{3}\right) \\
\Rightarrow \quad x(0) &= \frac{1}{3}\left((c_1 - c_2) + \frac{10}{3}\right) = 0 \\
\text{and} \quad y(0) &= c_1 + \frac{5}{3} = 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_1 &= -\frac{5}{3} \\
c_2 &= \frac{5}{3}
\end{aligned}
$$

and so

$$\begin{aligned} x(t) &= \left(-\frac{10}{9}\right)e^t + \frac{5}{3}te^t + \frac{10}{9} \\ y(t) &= -\frac{5}{3}e^t + 5te^t + \frac{5}{3}. \end{aligned}$$

6.

$$\begin{aligned} Dx + 4y &= 1 \Rightarrow D^2x + 4Dy = 0 \Rightarrow Dy = -\frac{1}{4}D^2x \\ Dy + x &= 2 \Rightarrow -\frac{1}{4}D^2x + x = 2 \Rightarrow D^2x - 4x = -8 \qquad (2) \end{aligned}$$

Also,

$$Dy + x = 2 \Rightarrow Dy = 2 - x \Rightarrow y = \int (2 - x)\, dt \qquad (3)$$

Eqn (2) is $x'' - 4x = -8$ and is solved using the method of undetermined coefficients to yield

$$x(t) = c_1e^{2t} + c_2e^{-2t} + 2.$$

Eqn (3) gives

$$\begin{aligned} y(t) &= \int \left(-c_1e^{2t} - c_2e^{-2t}\right) dt \\ &= -\frac{c_1e^{2t}}{2} + \frac{c_2e^{-2t}}{2} + c_3 \end{aligned}$$

Substituting $x(t)$ and $y(t)$ into $Dx - 4y = 1$ gives

$$4c_3 = 1 \Rightarrow c_3 = \frac{1}{4}$$

and so

$$\begin{aligned} x(t) &= c_1e^{2t} + c_2e^{-2t} + 2 \\ y(t) &= -\frac{c_1e^{2t}}{2} + \frac{c_2e^{-2t}}{2} + \frac{1}{4}. \end{aligned}$$