TWK2A
Cauchy-Euler DEs (Section 4.7)
Solutions

1. Using the substitution $u = y'$ and the assumption $u = x^m$ we obtain

$$\begin{aligned} xu' + u &= 0 \\ mx^m + x^m &= 0, \quad m = -1 \\ y' &= c_1 \frac{1}{x} \\ y &= c_1 \ln x + c_2 \end{aligned}$$

2. We assume $y = x^m$,

$$\begin{aligned} m(m-1)x^m + 5mx^m + 3x^m &= 0 \\ m^2 + 4m + 3 &= 0 \\ (m+1)(m+3) &= 0 \\ m_1 = -1, &\qquad m_3 = -3 \\ y &= c_1 x^{-1} + c_2 x^{-3} \end{aligned}$$

3. We first solve the associated homogenous differential equation using the assumption $y = x^m$

$$\begin{aligned} 2m(m-1) + 5m + 1 &= 0 \\ 2m^2 + 3m + 1 &= 0 \\ (2m+1)(m+1) &= 0 \\ m_1 = -\frac{1}{2}, &\qquad m_2 = -1 \\ y_c &= c_1 x^{-\frac{1}{2}} + c_2 x^{-1} \end{aligned}$$

Before applying variation of parameters we must bring the equation into standard form (i.e. divide by $2x^2$)

$$\begin{aligned}
W &= \begin{vmatrix} x^{-\frac{1}{2}} & x^{-1} \\ -\frac{1}{2}x^{-\frac{3}{2}} & -x^{-2} \end{vmatrix} = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{5}{2}} = -\frac{1}{2}x^{-\frac{5}{2}} \\
W_1 &= \begin{vmatrix} 0 & x^{-1} \\ \frac{1}{2} - \frac{1}{2}x^{-1} & -x^{-2} \end{vmatrix} = -\frac{1}{2}x^{-1} + \frac{1}{2}x^{-2} \\
W_2 &= \begin{vmatrix} x^{-\frac{1}{2}} & 0 \\ -\frac{1}{2}x^{-\frac{3}{2}} & \frac{1}{2} - \frac{1}{2}x^{-1} \end{vmatrix} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} \\
u_1' &= \frac{W_1}{W} = x^{\frac{3}{2}} - x^{\frac{1}{2}} \\
u_1 &= \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} \\
u_2' &= \frac{W_2}{W} = -x^2 + x \\
u_2 &= -\frac{1}{3}x^3 + \frac{1}{2}x^2
\end{aligned}$$

Thus the general solution is given by

$$y = c_1x^{-\frac{1}{2}} + c_2x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x.$$

4. Using $x = e^t$ we obtain

$$\begin{aligned}
y(x) &= y(e^t) \\
\frac{dy}{dt} &= \frac{dy}{dx}e^t \\
\frac{d^2y}{dt^2} &= \frac{d^2y}{dx^2}e^{2t} + e^t\frac{dy}{dx} = e^{2t}\frac{d^2y}{dx^2} + \frac{dy}{dt} \\
e^{2t}e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 9e^te^{-t}\frac{dy}{dt} + 25y &= 0 \\
\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25y &= 0 \\
y = e^{mt} &\Rightarrow \\
m^2 - 10m + 25 &= (m-5)^2 = 0 \\
y &= c_1e^{5t} + c_2te^{5t} \\
&= c_1x^5 + c_2x^5 \ln x
\end{aligned}$$

5.

$$\begin{aligned}
t &\equiv x+2\\
\frac{dy}{dt} &= \frac{dy}{dx}\\
t^2y''+ty'+y &= 0\\
y=t^m &\Rightarrow\\
m(m-1)+m+1 &= 0\\
m^2+1 &= 0\\
m_{1,2} &= \pm i\\
y &= c_1\cos(\ln t)+c_2\sin(\ln t)\\
&= c_1\cos(\ln(x+2))+c_2\sin(\ln(x+2))
\end{aligned}$$

6.

$$x^2y''+3xy'-4y=0$$

Substitute $y=x^m$; $y'=mx^{m-1}$; $y''=m(m-1)x^{m-1}$ in the above:

$$\begin{aligned}
& m(m-1)x^m+3mx^m-4x^m &= 0\\
\div x^m: \quad & m^2+2m-4 &= 0\\
& m=\frac{-2\pm\sqrt{4+16}}{2} &= -1\pm\sqrt{5}
\end{aligned}$$

Solution:

$$y(x)=c_1x^{-1+\sqrt{5}}+c_2x^{-1-\sqrt{5}}.$$

7.

$$x^3y'''+xy'-y=0$$

Substitute $y=x^m$; $y'=mx^{m-1}$; $y''=m(m-1)x^{m-2}$; $y'''=m(m-1)(m-2)x^{m-3}$ in the above:

$$\begin{aligned}
& m(m-1)(m-2)x^m+mx^m-x^m &= 0\\
\div x^m: \quad & m^3-3m^2+2m+m-1 &= 0\\
& m^3-3m^2+3m-1 &= 0\\
& (m-1)^3 &= 0\\
& \text{Roots:}\quad m_1=m_2=m_3 &= 1
\end{aligned}$$

Solution:

$$y(x)=c_1x+c_1x\ln x+c_2x(\ln x)^2$$

8.

$$x^2y'' - 2xy' + 2y = x^4c^x \tag{1}$$

For the associated homogeneous DE, we assume

$$y = x^m; \quad y' = mx^{m-1}; \quad y'' = m(m-1)x^{m-2}.$$

Hence,

$$\begin{aligned} & m(m-1)x^m - 2mx^m + 2x^m &=& \ 0 \\ \div x^m : \quad & m^2 - 3m + 2 &=& \ 0 \\ & (m-2)(m-1) &=& \ 0. \end{aligned}$$

The roots are

$$m_1 = 1; \ m_2 = 2$$

and so the complementary function is given by

$$y_c = c_1x + c_2x^2. \tag{2}$$

For a particular solution we assume

$$y_p = u_1(x)x + u_2(x)x^2. \tag{3}$$

Hence,

$$u_p' = u_1'x + u_2'x^2 + u_1 + 2u_2x.$$

Furthermore, we assume

$$u_1'x + u_2'x^2 = 0 \tag{4}$$

so that

$$y_p' = u_1 + 2u_2x \tag{5}$$

and so

$$y_p'' = u_1' + 2u_2'x + 2u_2. \tag{6}$$

Substitute (3), (5) and (6) into (1):

$$x^2u_1' + 2x^3u_2' + 2x^2u_2 - 2x(u_1 + 2u_2x) + 2u_1x + 2u_2x^2 = x^4e^x$$

$$\Rightarrow u_1' + 2xu_2' = x^2e^x \tag{7}$$

From (4) we have

$$u_1' = -xu_2'. \tag{8}$$

Substitute into (7):

$$-xu_2' + 2xu_1' = x^2e^x$$
$$\Rightarrow u_2' = xe^x \tag{9}$$

Integrate:

$$u_2 = xe^x - \int e^x\,dx = xe^x - e^x = (x-1)e^x. \tag{10}$$

Substitute (9) into (8):

$$u_1' = -x^2e^x.$$

Integrate:

$$\begin{aligned} u_1 &= -x^2e^x - \int(-2x)e^x\,dx \\ &= -x^2e^x + 2\left[xe^x - \int e^x\,dx\right] \\ &= -x^2e^x + 2xe^x - 2e^x \end{aligned}$$
$$\Rightarrow u_1 = \left(-x^2 + 2x - 2\right)e^x \tag{11}$$

Substitute (10) and (11) into (3):

$$y_p = (-x^3 + 2x^2 - 2x)e^x + (x^3 - x^2)e^x = (x^2 - 2x)e^x \tag{12}$$

The general solution follows from (2) and (12):

$$y = y_x + y_p = c_1x + c_2x^2 + (x^2 - 2x)e^x.$$

9.

$$x^2y'' - 3xy' + 4y = 0 \tag{13}$$

with initial conditions

$$y(1) = 5; \quad y'(1) = 3 \tag{14}$$

We substitute

$$y = x^m; \quad y' = mx^{m-1}; \quad y'' = m(m-1)x^{m-1}$$

into (13):

$$\begin{aligned} m(m-1)x^m - 3mx^m + 4x^m &= 0 \\ \div x^m: \quad m^2 - 4m + 4 &= 0 \\ (m-2)^2 &= 0 \end{aligned}$$

The roots therefore are

$$m_1 = m_2 = 2$$

and the solution is given by

$$y = c_1 x^2 + c_2 x^2 \ln x.$$

Hence

$$y' = 2c_1 x + 2c_2 x \ln x + c_2 x.$$

Introduce (14):

$$\begin{aligned} &y(1) = c_1 = 5 \\ &y'(1) = 2c_1 + c_2 = 3; \quad c_2 = 3 - 2c_1 = -7 \end{aligned}$$

Thus

$$y = 5x^2 - 7x^2 \ln x$$

10. We have

$$xy'' - 3y' = 0$$

or

$$x^2 y'' - 3xy' = 0. \tag{15}$$

Thus, the DE is a Cauchy-Euler DE and we substitute

$$y = x^m; \quad y' = mx^{m-1}; \quad y'' = m(m-1)x^{m-2}$$

into (15):

$$m(m-1)x^m - 3mx^m = 0.$$

On $(0, \infty)$ we have

$$\begin{aligned} m(m-1) - 3m &= 0 \\ m(m-4) &= 0 \\ m &= 0 \text{ or } 4. \end{aligned}$$

Hence, the general solution is

$$y(x) = c_1 + c_2 x^4.$$

11.

$$x^2y'' - 5xy' + 8y = 8x^6 \tag{16}$$

with initial-values

$$y\left(\frac{1}{2}\right) = 0; \quad y'\left(\frac{1}{2}\right) = 0 \tag{17}$$

The DE is clearly Cauchy-Euler and we substitute

$$y = x^m; \quad y' = mx^{m-1}; \quad y'' = m(m-1)x^{m-2}$$

into the homogeneous DE associated with (16):

$$m(m-1)x^m - 5mx^m + 8x^m = 0$$

On $(0, \infty)$ we have

$$\begin{aligned} m(m-1) - 5m + 8 &= 0 \\ m^2 - 6m + 8 &= 0 \\ (m-2)(m-4) &= 0 \\ m &= 2 \text{ or } 4 \end{aligned}$$

Thus, the complementary function is

$$y_c = c_1x^2 + c_2x^4 \tag{18}$$

Write (16) in standard form:

$$y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 8x^4 \tag{19}$$

We assume a particular solution of the form

$$y_p = u_1(x)x^2 + u_2(x)x^4 \tag{20}$$

Hence,

$$\begin{aligned} W &= \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = 4x^5 - 2x^5 = 2x^5 \\ W_1 &= \begin{vmatrix} 0 & x^4 \\ 9x^4 & 4x^3 \end{vmatrix} = -8x^8 \\ W_2 &= \begin{vmatrix} x^2 & 0 \\ 2x & 8x^4 \end{vmatrix} = 8x^6 \end{aligned}$$

and so

$$\begin{aligned} u_1' &= \frac{W_1}{W} = -4x^3 \\ u_2' &= \frac{W_2}{W} = 4x \end{aligned}$$

Integration yields

$$\begin{aligned} u_1 &= -x^4 \\ u_2 &= 2x^2 \end{aligned}$$

Substitute into (20):

$$y_p = -x^4x^2 + 2x^2x^4 = x^6$$

The general solution is thus

$$y(x) = y_c + y_p = c_1x^2 + c_2x^4 + x^6. \tag{21}$$

Hence,

$$y' = 2c_1x + 4c_2x^3 + 6x^5. \tag{22}$$

Substitute $y\left(\frac{1}{2}\right) = 0$ into (21):

$$\begin{aligned} \frac{1}{4}c_1 + \frac{1}{16}c_2 + \frac{1}{64} &= 0 \\ 16c_1 + 4c_2 &= 1 \end{aligned} \tag{23}$$

Ssubstitute $y'\left(\frac{1}{2}\right) = 0$ into (22):

$$\begin{aligned} c_1 + \frac{1}{2}c_2 + \frac{6}{32} &= 0 \\ 16c_1 + 8c_2 &= -3 \end{aligned} \tag{24}$$

(24) – (23):

$$4c_2 = -2; \quad c_2 = -\frac{1}{2}.$$

Substitute into (23):

$$c_1 = -\frac{1}{16}(1 + 4c_2) = \frac{1}{16}.$$

And so the solution to the IVP is

$$y(x) = x^6 - \frac{1}{2}x^4 + \frac{1}{16}x^2.$$

12. Since $1-\mathbf{i}$ is a root, so must $1+\mathbf{i}$ be a root. The auxiliary equation is thus

$$\begin{aligned}
(m-2)[m-(1-\mathbf{i})][m-(1+\mathbf{i})] &= 0 \\
(m-2)[m^2-(1-\mathbf{i}+1+\mathbf{i})m+(1+1)] &= 0 \\
(m-2)(m^2-2m+2) &= 0 \\
m^3-4m^2+6m-4 &= 0 \qquad (25)
\end{aligned}$$

It would appear that the Cauchy-Euler DE is of 3rd order, and so has the form

$$ax^3y''' + bx^2y'' + cxy' + ey = 0. \qquad (26)$$

We assume a solution of the form $y = x^m$. Hence,

$$y' = mx^{m-1}; \quad y'' = m(m-1)x^{m-2}; \quad y''' = m(m-1)(m-2)x^{m-3}.$$

Substitute into (26):

$$am(m-1)(m-2)x^m + bm(m-1)x^m + cmx^m + ex^m = 0.$$

On $(0,\infty)$ we have

$$am(m-1)(m-2) + bm(m-1) + cm + e = 0.$$

Simplify:

$$am^3 + (b-3a)m^2 + (2a-b+c)m + e = 0. \qquad (27)$$

Equations (25) and (27) are characteristic equations of the same Cauchy-Euler DE and so their coefficients must be the same:

$$\begin{aligned}
a &= 1 \\
b-3a &= -4 \Rightarrow b = 3a-4 = -1 \\
2a-b+c &= 6 \Rightarrow c = 6-2a+b = 3 \\
e &= -4
\end{aligned}$$

Equation (26) thus becomes

$$x^3y''' - x^2y'' + 3xy' - 4y = 0.$$

13.

$$x^2y'' - xy' + y = 2x :$$

$$\begin{aligned}
x^2y'' - xy' + y &= 0 \\
&\Rightarrow\ m^2 - 2m + 1 = 0 \ \ (\text{assuming } y = x^m) \\
&\Rightarrow\ m_1 = 1, m_2 = 1
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\ y_c(x) = c_1x + c_2x\ln x \\
&\Rightarrow\ y_1 = x \ \text{ and } \ y_2 = x\ln x \\
&\Rightarrow\ y_1' = 1 \ \text{ and } \ y_2' = 1 + \ln x
\end{aligned}$$

Standard form of DE: $$y'' - \frac{y'}{x} + \frac{y}{x^2} = \frac{2}{x}$$

$$\Rightarrow\ f(x) = \frac{2}{x}$$

$$\begin{aligned}
\text{Hence, } W_1 &= \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & 1+\ln x \end{vmatrix} = -2\ln x \\
W_2 &= \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix} = 2 \\
W &= \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x
\end{aligned}$$

$$\begin{aligned}
\text{and so } u_1' &= \frac{W_1}{W} = \frac{-2\ln x}{x} \Rightarrow u_1 = -(\ln x)^2 \\
u_2' &= \frac{W_2}{W} = \frac{2}{x} \Rightarrow u_2 = 2\ln x
\end{aligned}$$

$$\begin{aligned}
y_p &= u_1y_1 + u_2y_2 \\
&= -(\ln x)^2 x + (2\ln x)\, x\ln x \\
&= -x(\ln x)^2 + 2x(\ln x)^2 \\
&= x(\ln x)^2
\end{aligned}$$

$$\Rightarrow y(x) = c_1x + c_2x\ln x + x(\ln x)^2 .$$

$\left[\text{Note: } \int \frac{\ln x}{x}dx \ \text{ is evaluated using the substitution } u = \ln x \ \left(\Rightarrow du = \frac{dx}{x}\right).\right]$