TWK2A
Variation of parameters (Section 4.6)
Solutions

1.

$$y'' + y = \sin x$$

Homogeneous equation:

$$\begin{aligned} y'' + y = 0 \quad &\Rightarrow \quad m^2 + 1 = 0 \\ &\Rightarrow \quad m_1 = \mathbf{i}, \qquad m_2 = -\mathbf{i} \end{aligned}$$

So $y_c = c_1 \cos(x) + c_2 \sin(x)$
For particular solution:

$$\begin{aligned} W &= \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1 \\ W_1 &= \begin{vmatrix} 0 & \sin(x) \\ \sin(x) & \cos(x) \end{vmatrix} = -\sin^2(x) \\ W_2 &= \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sin(x) \end{vmatrix} = \cos(x)\sin(x) \end{aligned}$$

So

$$\begin{aligned} u_1' &= -\sin^2(x) \Rightarrow u_1 = -\frac{x}{2} - \frac{1}{4}\sin(2x) = \frac{1}{2}\sin(x)\cos(x) - \frac{1}{2}x \\ u_2' &= \cos(x)\sin(x) \Rightarrow u_2 = -\frac{1}{2}\cos^2(x) \end{aligned}$$

Hence

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 = \frac{1}{2}\sin(x)\cos^2(x) - \frac{1}{2}x\cos(x) - \frac{1}{2}\cos^2(x)\sin(x) \\ &= -\frac{1}{2}x\cos(x) \end{aligned}$$

So

$$y(x) = y_c + y_p = c_1\cos(x) + c_2\sin(x) - \frac{1}{2}x\cos(x)$$

(valid for $-\infty < x < \infty$).

2.

$$y'' - y = \cosh\left(\frac{x}{2}\right) = \frac{e^{\frac{x}{2}}}{2} + \frac{e^{-\frac{x}{2}}}{2}$$

Homogeneous equation:

$$\begin{aligned} y'' - y = 0 \;&\Rightarrow\; m^2 - 1 = 0 \\ &\Rightarrow\; m = \pm 1 \\ &\Rightarrow\; y_c = c_1 e^x + c_2 e^{-x} \end{aligned}$$

Particular solution:

$$\begin{aligned} W &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \\ W_1 &= \begin{vmatrix} 0 & e^{-x} \\ \cosh\left(\frac{x}{2}\right) & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{2}\left(e^{\frac{x}{2}} + e^{\frac{-x}{2}}\right) = -\frac{e^{\frac{-x}{2}}}{2} - \frac{e^{\frac{-3x}{2}}}{2} \\ W_2 &= \begin{vmatrix} e^x & 0 \\ e^x & \cosh\left(\frac{x}{2}\right) \end{vmatrix} = \frac{e^{\frac{3x}{2}}}{2} + \frac{e^{\frac{x}{2}}}{2} \end{aligned}$$

So

$$\begin{aligned} u_1' = \frac{W_1}{W} &= \frac{e^{\frac{-x}{2}}}{4} + \frac{e^{\frac{-3}{2}}}{4} \\ \Rightarrow u_1 &= -\frac{e^{\frac{-x}{2}}}{2} - \frac{e^{\frac{-3x}{2}}}{6} \\ u_2' = \frac{W_2}{W} &= -\frac{e^{\frac{3x}{2}}}{4} - \frac{e^{\frac{x}{2}}}{4} \\ \Rightarrow u_2 &= -\frac{e^{\frac{3x}{2}}}{6} - \frac{e^{\frac{x}{2}}}{2} \end{aligned}$$

Thus

$$\begin{aligned} y(x) = y_c + y_p &= y_c + u_1 y_1 + u_2 y_2 \\ &= c_1 e^x + c_2 e^{-x} - \frac{e^{\frac{x}{2}}}{2} - \frac{e^{\frac{-x}{2}}}{6} - \frac{e^{\frac{x}{2}}}{6} - \frac{e^{\frac{-x}{2}}}{2} \\ &= c_1 e^x + c_2 e^{-x} - \frac{2e^{\frac{x}{2}}}{3} - \frac{2e^{\frac{-x}{2}}}{3} \end{aligned}$$

(valid for all x).

3.

$$4y'' - y = xe^{\frac{x}{2}} \qquad y(0) = 1, \qquad y'(0) = 0$$

Homogeneous case:

$$\begin{aligned} 4y'' - y = 0 \quad &\Rightarrow \quad 4m^2 - 1 = 0 \\ &\Rightarrow \quad m = \pm\frac{1}{2} \end{aligned}$$

So

$$y_c = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{-x}{2}}.$$

Particular solution:

$$\begin{aligned} W &= \begin{vmatrix} e^{\frac{x}{2}} & e^{\frac{-x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & -\frac{1}{2}e^{\frac{-x}{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1 \\ W_1 &= \begin{vmatrix} 0 & e^{\frac{-x}{2}} \\ \frac{xe^{\frac{x}{2}}}{4} & -\frac{1}{2}e^{\frac{-x}{2}} \end{vmatrix} = -\frac{x}{4} \\ W_2 &= \begin{vmatrix} e^{\frac{x}{2}} & 0 \\ \frac{1}{2}e^{\frac{x}{2}} & \frac{xe^{\frac{x}{2}}}{4} \end{vmatrix} = \frac{xe^x}{4} \end{aligned}$$

Hence,

$$\begin{aligned} y(x) &= y_c + y_p \\ &= y_c + u_1 y_1 + u_2 y_2 \\ &= c_1 e^{\frac{x}{2}} + c_2 e^{\frac{-x}{2}} + \frac{x^2}{8} e^{\frac{x}{2}} + e^{\frac{-x}{2}} \frac{(-xe^x + e^x)}{4} \\ &= c_1 e^{\frac{x}{2}} + c_2 e^{\frac{-x}{2}} + \frac{x^2 e^{\frac{x}{2}}}{8} - \frac{xe^{\frac{x}{2}}}{4} + \frac{e^{\frac{x}{2}}}{4} \end{aligned}$$

Now,

$$\begin{aligned} y(0) &= c_1 + c_2 + \frac{1}{4} = 1 \\ y'(0) &= \frac{c_1}{2} - \frac{c_2}{2} - \frac{1}{8} = 0 \end{aligned}$$

So

$$\begin{aligned} c_1 + c_2 &= \frac{3}{4} \\ c_1 - c_2 &= \frac{1}{4} \\ \Rightarrow c_1 &= \frac{1}{2},\ c_2 = \frac{1}{4} \end{aligned}$$

Thus,

$$\begin{aligned} y(x) &= \frac{e^{\frac{x}{2}}}{2} + \frac{e^{\frac{-x}{2}}}{4} + \frac{x^2 e^{\frac{x}{2}}}{8} - \frac{x e^{\frac{x}{2}}}{4} + \frac{e^{\frac{x}{2}}}{4} \\ &= \frac{3e^{\frac{x}{2}}}{4} + \frac{e^{\frac{-x}{2}}}{4} + \frac{x^2 e^{\frac{x}{2}}}{8} - \frac{x e^{\frac{x}{2}}}{4} \end{aligned}$$

4.

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = x^{\frac{x}{2}}$$

Standard form:

$$y'' + \frac{1}{x} y' + \frac{\left(x^2 - \frac{1}{4}\right)}{x^2} y = x^{-\frac{1}{2}}$$

$$\begin{aligned} W &= \begin{vmatrix} x^{\frac{-1}{2}} \cos x & x^{\frac{-1}{2}} \sin x \\ -\frac{1}{2} x^{\frac{-3}{2}} \cos x - x^{\frac{-1}{2}} \sin x & -\frac{1}{2} x^{\frac{-3}{2}} \sin x + x^{\frac{-1}{2}} \cos x \end{vmatrix} \\ &= -\frac{1}{2} x^{-2} \cos x \sin x + x^{-1} \cos^2 x - \left(-\frac{1}{2} x^{-2} \cos x \sin x - x^{-1} \sin^2 x\right) \\ &= x^{-1} \left(\cos^2 x + \sin^2 x\right) = \frac{1}{x} \\ W_1 &= \begin{vmatrix} 0 & x^{\frac{-1}{2}} \sin x \\ x^{\frac{-1}{2}} & -\frac{1}{2} x^{\frac{-3}{2}} \sin x + x^{\frac{-1}{2}} \cos x \end{vmatrix} = -x^{-1} \sin x \\ W_2 &= \begin{vmatrix} x^{\frac{-1}{2}} \cos x & 0 \\ -\frac{1}{2} x^{\frac{-3}{2}} \cos x - x^{\frac{-1}{2}} \sin x & x^{\frac{-1}{2}} \end{vmatrix} = x^{-1} \cos x \end{aligned}$$

Hence,

$$\begin{aligned} u_1' &= \frac{W_1}{W} = \frac{-x^{-1} \sin x}{x^{-1}} = -\sin x \Rightarrow u_1 = \cos(x) \\ u_2' &= \frac{W_2}{W} = \frac{x^{-1} \cos x}{x^{-1}} = \cos x \Rightarrow u_2 = \sin x \end{aligned}$$

So

$$\begin{aligned} y(x) &= y_c + y_p = y_c + u_1 y_1 + u_2 y_2 \\ &= c_1 x^{\frac{1}{2}} \cos x + c_2 x^{\frac{-1}{2}} \sin x + x^{\frac{-1}{2}} \cos^2 x + x^{\frac{-1}{2}} \sin^2 x \\ &= c_1 x^{\frac{-1}{2}} \cos x + c_2 x^{\frac{-1}{2}} \sin x + x^{\frac{-1}{2}} \end{aligned}$$

5.

$$y'' - 2y' + y = 4x^2 - 3 + \frac{e^x}{x}$$

Homogeneous case:

$$\begin{aligned} & y'' - 2y' + y = 0 \\ \Rightarrow \quad & m^2 - 2m + 1 = 0 \\ \Rightarrow \quad & m_1 = 1, m_2 = 1 \quad \text{(repeated real roots)} \end{aligned}$$

So

$$y_c = c_1 e^x + c_2 x e^x.$$

Now, we have

$$L_y = y'' - 2y' + y = f(x) + g(x)$$

where $f(x) = 4x^2 - 3$ and $g(x) = \frac{e^x}{x}$.
Note that, if y_{p_1} is a particular solution of $L_y = f(x)$ and y_{p_2} is a particular solution of $L_y = g(x)$, then $y_{p_1} + y_{p_2}$ is a particular solution of $L_y = f(x) + g(x)$.
We use method of undetermined coefficients to solve

$$y'' - 2y' + y = 4x^2 - 3$$

and variation of parameters to solve

$$y'' - 2y' + y = \frac{e^x}{x}$$

to get particular solutions.
For

$$y'' - 2y' + y = 4x^2 - 3$$

assume the form for the solution is $Ax^2 + Bx + c$.
So $y'' = 2A, y' = 2Ax + B$ and $y = Ax^2 + Bx + C$ which gives

$$\begin{aligned} & (2A) - 2(2Ax + B) + (Ax^2 + Bx + C) = 4x^2 - 3 \\ \Rightarrow \quad & Ax^2 + (B - 4A)x + (2A + C - 2B) = 4x^2 - 3 \\ \Rightarrow \quad & A = 4, \quad B = 16, \quad C = 21 \end{aligned}$$

Hence, $y_{p_1} = 4x^2 + 16x + 21$.

For

$$y'' - 2y' + y = \frac{e^x}{x}$$

$$\begin{aligned} W &= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x} \\ W_1 &= \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & xe^x + e^x \end{vmatrix} = -e^{2x} \\ W_2 &= \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix} = \frac{e^{2x}}{x} \\ u_1' &= \frac{W_1}{W} = -1 \;\Rightarrow u_1 = -x \\ u_2' &= \frac{W_2}{W} = \frac{e^{2x}}{e^{2x}x} = \frac{1}{x} \;\Rightarrow u_2 = \ln|x| \end{aligned}$$

So

$$\begin{aligned} y_{p_2} &= u_1 y_1 + u_2 y_2 \\ &= -xe^x + xe^x \ln|x| \end{aligned}$$

General solution:

$$\begin{aligned} y(x) &= y_c + y_{p_1} + y_{p_2} \\ &= c_1 a^x + c_2 xe^x + 4x^2 + 16x + 21 - xe^x + xe^x \ln|x|. \end{aligned}$$

6.

$$y'' + y = \sec x \tan x \tag{1}$$

The complementary function is found by solving the auxiliary equation

$$m^2 + 1 = 0.$$

The roots are $\pm\mathbf{i}$, and hence

$$y_c = c_1 \cos x + c_2 \sin x. \tag{2}$$

We now assume a particular solution of the form

$$y_p = u_1(x) \cos x + u_2(x) \sin x. \tag{3}$$

Then follows that

$$y_p' = u_1' \cos x + u_2' \sin x - u_1 \sin x + u_2 \cos x.$$

If we make the assumption that

$$u_1' \cos x + u_2' \sin x = 0, \tag{4}$$

the first derivative simplifies to

$$y_p' = -u_1 \sin x + u_2 \cos x.$$

From further differentiation we have

$$y_p'' = -u_1' \sin x + u_2' \cos x - u_1 \cos x - u_2 \sin x. \tag{5}$$

We substitute (3) and (5) in (1):

$$y_p'' + y_p' = -u_1' \sin x + u_2' \cos x = \sec x \tan x \tag{6}$$

We now solve (4) and (6) for u_1 and u_2. From (4) we have

$$u_1' = -u_2' \tan x. \tag{7}$$

Substitute (7) in (6):

$$\begin{array}{lrcl} & u_2' \sin x \tan x + u_2' \cos x & = & \sec x \tan x \\ \times \cos x : & (\sin^2 x + \cos^2 x) u_2' & = & \tan x \end{array}$$

$$u_2' = \tan x = \frac{\sin x}{\cos x}. \tag{8}$$

Integrate w.r.t. x:

$$u_2 = -\ln|\cos x|. \tag{9}$$

Substitute (8) in (7):

$$u_1' = -\tan^2 x = 1 - \sec^2 x.$$

Integrate w.r.t. x:

$$u_1 = x - \tan x \tag{10}$$

Substituting (9) and (10) in (3), we have the particular solution

$$y_p = (x - \tan x)\cos x - \sin x \ln|\cos x| = x\cos x - \sin x - \sin x \ln|\cos x|. \tag{11}$$

The general solution follows from (2) and (11):

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + x\cos x - \sin x - \sin x \ln|\cos x| \\ &= c_1 \cos x + c_3 \sin x + x\cos x - \sin x \ln|\cos x|. \end{aligned}$$

7.

$$y'' - 9y = \frac{9x}{e^{3x}}. \tag{12}$$

The complementary function is found by solving the auxiliary equation

$$m^2 - 9 = 0.$$

which has roots ± 3, and so

$$y_c = c_1 e^{3x} + c_2 e^{-3x} \tag{13}$$

We assume a particular solution of the form

$$y_p = u_1(x)e^{3x} + u_2(x)e^{-3x}. \tag{14}$$

Hence,

$$y_p' = u_1' e^{3x} + u_2' e^{-3x} + 3u_1 e^{3x} - 3u_2 e^{-3x}.$$

Furthermore, we assume

$$u_1' e^{3x} + u_2' e^{-3x} = 0, \tag{15}$$

so that

$$y_p' = 3u_1 e^{3x} - 3u_2 e^{-3x}$$

and

$$y_p'' = 3u_1'e^{3x} - 3u_2'e^{-3x} + 9u_1e^{3x} + 9u_2e^{-3x}. \tag{16}$$

Substitute (14) and (16) into (12):

$$\begin{aligned} y_p'' - 9y_p' &= 3u_1'e^{3x} - 3u_2'e^{-3x} = \frac{9x}{e^{3x}} \\ \Rightarrow \quad & u_1'e^{3x} - u_2'e^{-3x} = 3xe^{-3x}. \end{aligned} \tag{17}$$

We solve (15) and (17) for the u_i. From (15 we have

$$u_2' = -u_1'e^{6x}. \tag{18}$$

Substitute (18) into (17):

$$u_1'e^{3x} + u_1'e^{3x} = 3xe^{-3x} \Rightarrow u_1' = \frac{3}{2}xe^{-6x} \tag{19}$$

Integrate w.r.t. x using integration by parts:

$$\begin{aligned} u_1 &= \frac{3}{2}\int xe^{-6x}\,dx \\ &= \frac{3}{2}\left[-\frac{x}{6}e^{-6x} - \int \frac{1}{(-6)}e^{-6x}\,dx\right] \\ &= \frac{3}{2}\left[-\frac{x}{6}e^{-6x} - \frac{1}{36}e^{-6x}\right]. \end{aligned}$$

Hence,

$$u_1 = -\frac{1}{24}(6x+1)e^{-6x}. \tag{20}$$

Substitute (19) into (18):

$$u_2' = -\frac{3}{2}x.$$

Integrate w.r.t. x:

$$u_2 = -\frac{3}{4}x^2. \tag{21}$$

We substitute (20) and (21 into (14) to find a particular solution:

$$y_p = -\frac{1}{24}(6x+1)e^{-6x}e^{3x} - \frac{3}{4}x^2e^{-3x} = -\frac{1}{24}(18x^2+6x+1)e^{-3x}. \tag{22}$$

The general solution follows from (13) and (22):

$$\begin{aligned} y &= y_c + y_p \\ &= c_1e^{3x} + c_2e^{-3x} - \frac{1}{24}(18x^2 + 6x + 1)e^{-3x} \\ &= c_1e^{3x} + c_3e^{-3x} - \frac{1}{4}(3x + 1)xe^{-3x}. \end{aligned}$$

8.

$$2y'' + y' - y = x + 1 \tag{23}$$

with the initial conditions

$$y(0) = 1; \quad y'(0) = 0. \tag{24}$$

The complementary function follows from the auxiliary equation

$$2m^2 + m - 1 = 0 \Rightarrow (2m - 1)(m + 1) = 0$$

The roots are $m_1 = \frac{1}{2}$; $m_2 = -1$ and we have

$$y_c = c_1e^{\frac{1}{2}x} + c_2e^{-x}. \tag{25}$$

We assume a particular solution of the form

$$y_p = u_1(x)e^{\frac{1}{2}x} + u_2(x)e^{-x}. \tag{26}$$

Then follows

$$y_p' = u_1'e^{\frac{1}{2}x} + u_2'e^{-x} + \frac{1}{2}u_1(x)e^{\frac{1}{2}x} - u_2e^{-x}.$$

If we make the further assumption that

$$u_1'e^{\frac{1}{2}x} + u_2'e^{-x} = 0, \tag{27}$$

we have

$$y_p' = \frac{1}{2}u_1e^{\frac{1}{2}x} - u_2e^{-x} \tag{28}$$

and

$$y_p'' = \frac{1}{2}u_1'e^{\frac{1}{2}x} - u_2'e^{-x} + \frac{1}{4}u_1e^{\frac{1}{2}x} + u_2e^{-x}. \tag{29}$$

Substitute (26), (28) and (29) into (23):

$$2y_p'' + y_p' - y_p = u_1' e^{\frac{1}{2}x} - 2u_2' e^{-x} = x + 1. \tag{30}$$

We now solve (27) and (30) for the u_i. From (27) we have

$$u_2' = -u_1' e^{\frac{3}{2}x}. \tag{31}$$

Substitute (31) into (30):

$$u_1' e^{\frac{1}{2}x} + 2u_1' e^{\frac{1}{2}x} = x + 1 \Rightarrow u_1' = \frac{1}{3}(x+1)e^{-\frac{1}{2}x} \tag{32}$$

Integrate by parts:

$$\begin{aligned} u_1 &= \frac{1}{3}\int e^{-\frac{1}{2}x}\,dx + \frac{1}{3}\int xe^{-\frac{1}{2}x}\,dx \\ &= -\frac{2}{3}e^{-\frac{1}{2}x} + \frac{1}{3}\left[-2xe^{-\frac{1}{2}x} - \int (-2)e^{-\frac{1}{2}x}\,dx\right] \\ &= -\frac{2}{3}e^{-\frac{1}{2}x} + \frac{1}{3}\left[-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}\right]. \end{aligned}$$

and so

$$u_1 = -\left(\frac{2}{3}x + 2\right)e^{-\frac{1}{2}x}. \tag{33}$$

Substitute (32) into (31):

$$u_2' = -\frac{1}{3}(x+1)e^x.$$

Integrate w.r.t. x:

$$\begin{aligned} u_2 &= -\frac{1}{3}\int xe^x\,dx - \frac{1}{3}\int e^x\,dx \\ &= -\frac{1}{3}\left[xe^x - \int e^x\,dx\right] - \frac{1}{3}e^x \\ &= -\frac{1}{3}\left[xe^x - e^x\right] - \frac{1}{3}e^x \\ &= -\frac{1}{3}xe^x \end{aligned} \tag{34}$$

We substitute (33) and (34) into (26) to obtain a particular solution:

$$\begin{aligned} y_p &= -\left(\frac{2}{3}x+2\right)e^{-\frac{1}{2}x}e^{\frac{1}{2}x} - \frac{1}{3}xe^x e^{-x} \\ &= -\frac{2}{3}x - 2 - \frac{1}{3}x \\ &= -(x+2) \end{aligned} \tag{35}$$

The general solution follows from (25) and (35):

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-x} - (x+2) \tag{36}$$

We now introduce the initial conditions. Firstly, we differentiate (14) to obtain

$$y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} - c_2 e^{-x} - 1. \tag{37}$$

Now introduce (24):

$$\begin{aligned} y(0) &= c_1 + c_2 - 2 = 1 \\ y'(0) &= \frac{1}{2}c_1 - c_2 - 1 = 0 \end{aligned}$$

We readily solve these equations to obtain

$$c_1 = \frac{8}{3}; \quad c_2 = \frac{1}{3}.$$

The solution (36) now becomes

$$y = \frac{1}{3}\left(8e^{\frac{1}{2}x} + e^{-x}\right) - (x+2).$$

9.

$$y''' + 4y' = \sec 2x \tag{38}$$

The complementary function follows from the auxiliary equation

$$m^3 + 4m = 0 \Rightarrow m(m^2+4) = 0$$

which has roots $0, \pm 2\mathbf{i}$ and so

$$y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x. \tag{39}$$

We assume a particular solution of the form

$$y_p = u_1(x) + u_2(x)\cos 2x + u_3(x)\sin 2x. \tag{40}$$

Hence,

$$y_p' = u_1' + u_2'\cos 2x + u_3'\sin 2x - 2u_2\sin 2x + 2u_3\cos 2x.$$

Furthermore, we assume that

$$u_1' + u_2'\cos 2x + u_3'\sin 2x = 0, \tag{41}$$

so that

$$y_p' = -2u_2\sin 2x + 2u_3\cos 2x \tag{42}$$

and

$$y_p'' = -2u_2'\sin 2x + 2u_3'\cos 2x - 4u_2\cos 2x - 4u_3\sin 2x.$$

Similar to (41) we assume

$$-2u_2'\sin 2x + 2u_3'\cos 2x = 0 \tag{43}$$

so that

$$y_p'' = -4u_2\cos 2x - 4u_3\sin 2x \tag{44}$$

and

$$y_p''' = -4u_2'\cos 2x - 4u_3'\sin 2x + 8u_2\sin 2x - 8u_3\cos 2x. \tag{45}$$

We substitute (42) and (45) into (38):

$$y_p''' + 4y_p' = -4u_2'\cos 2x - 4u_3'\sin 2x. \tag{46}$$

In (41), (43) and (46) we have three simultaneous equations which may be solved for the three u_i. From (43) we have

$$u_3' = u_2'\tan 2x. \tag{47}$$

Substitute (47) into (46):

$$\begin{array}{rrcl} & -4u_2'\cos 2x - 4u_2'\tan 2x\sin 2x & = & \sec 2x \\ \times\cos 2x: & -4u_2'\left(\cos^2 2x + \sin^2 2x\right) & = & 1 \end{array}$$

and so

$$u_2' = -\frac{1}{4}. \tag{48}$$

Integrate w.r.t. x:

$$u_2 = -\frac{1}{4}x. \tag{49}$$

Substitute (48) into (47):

$$u_3' = -\frac{1}{4}\tan 2x = -\frac{1}{4}\frac{\sin 2x}{\cos 2x}. \tag{50}$$

Integrate w.r.t. x:

$$u_3 = \frac{1}{8}\ln|\cos 2x|. \tag{51}$$

Substitute (48) and (50) into (41):

$$u_1' - \frac{1}{4}\cos 2x - \frac{1}{4}\tan 2x \sin 2x = 0.$$

Hence,

$$u_1' = \frac{1}{4}\cos 2x + \frac{1}{4}\frac{\sin^2 2x}{\cos 2x} = \frac{1}{4}\frac{\cos^2 2x + \sin^2 2x}{\cos 2x} = \frac{1}{4}\sec 2x.$$

Integrate w.r.t. x:

$$u_1 = \frac{1}{8}\ln|\sec 2x + \tan 2x|. \tag{52}$$

We substitute (49), (51) and (52) into (40):

$$y_p = \frac{1}{8}\ln|\sec 2x + \tan 2x| - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x \ln|\cos 2x|. \tag{53}$$

The general solution follows from (39) and (53):

$$y = c_1 + c_2\cos 2x + c_3\sin 2x + \frac{1}{8}\ln|\sec 2x + \tan 2x| - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x\ln|\cos 2x|.$$

This seems to be a rather long-winded approach. Let us use Wronskians instead to set up the relevant equations for the u's. Say

$$y_1 = 1, y_2 = \cos 2x, y_3 = \sin 2x.$$

Using

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} = AEI - AFH - DBI + DCH + GBF - GCE.$$

we find

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix} = 8$$

$$\begin{aligned} W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix} &= 2\sec 2x\cos^2 2x + 2\sec 2x \sin^2 2x \\ &= 2\sec 2x \end{aligned}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix} = -2\cos 2x \sec 2x = -2$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f \end{vmatrix} = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix} = -2\sin 2x \sec 2x$$

Hence,

$$\begin{aligned} u_1' &= \frac{W_1}{W} = \frac{2\sec 2x}{8} = \frac{\sec 2x}{4} \\ u_2' &= \frac{W_2}{W} = \frac{-2}{8} = -\frac{1}{4} \\ u_3' &= \frac{W_3}{W} = \frac{-2\sin 2x \sec 2x}{8} = -\frac{\sin 2x}{4\cos 2x} = -\frac{\tan 2x}{4} \end{aligned}$$

which seems to be a much easier approach.

10.

$$y'' + y = \tan x.$$

Auxiliary equation:

$$\begin{aligned} m^2 + 1 &= 0 \\ \Rightarrow m &= \pm i \end{aligned}$$

Complementary function:

$$y_c = c_1 \cos x + c_2 \sin x.$$

Assume a particular solution of the form

$$y_p = u_1(x) \cos x + u_2(x) \sin x. \tag{54}$$

Hence,

$$\begin{aligned}
W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1 \\
W_1 &= \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\sin x \tan x \\
W_2 &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \tan x = \sin x
\end{aligned}$$

and so

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = -\sin x \tan x \\
u_2' &= \frac{W_2}{W} = \sin x.
\end{aligned}$$

Integration yields

$$\begin{aligned}
u_1 &= -\int \sin x \tan x \, dx \\
&= -\left[-\cos x \tan x - \int (-\cos x) \sec^2 x \, dx \right] \\
&= \cos x \tan x - \int \sec x \, dx \\
&= \sin x - \ln |\sec x + \tan x|.
\end{aligned}$$

and

$$u_2 = \int \sin x \, dx = -\cos x.$$

Substitute this expression into (54):

$$
\begin{aligned}
y_p &= (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x \\
&= -\cos x \ln|\sec x + \tan x|.
\end{aligned}
$$

The general solution is thus

$$
\begin{aligned}
y &= y_c + y_p \\
&= (c_1 - \ln|\sec x + \tan x|)\cos x + c_2 \sin x.
\end{aligned}
$$

11.

$$y'' - 4y' + 4y = \left(12x^2 - 6x\right)e^{2x} \tag{55}$$

with initial-values

$$y(0) = 1; \quad y'(0) = 0. \tag{56}$$

Auxiliary equation:

$$
\begin{aligned}
m^2 - 4m + 4 &= 0 \Rightarrow (m-2)^2 = 0 \\
\Rightarrow m &= 2 \ \ \text{(twice)}.
\end{aligned}
$$

Complementary function:

$$y_c = c_1 e^{2x} + c_2 x e^{2x}. \tag{57}$$

Assume a particular solution of the form

$$y_p = u_1(x)e^{2x} + u_2(x)xe^{2x}. \tag{58}$$

Hence,

$$
\begin{aligned}
W &= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} = (1+2x)e^{4x} - 2xe^{4x} = e^{4x} \\
W_1 &= \begin{vmatrix} 0 & xe^{2x} \\ (12x^2-6x)e^{2x} & (1+2x)e^{2x} \end{vmatrix} = -(12x^3 - 6x^2)e^{4x} \\
W_2 &= \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (12x^2-6x)e^{2x} \end{vmatrix} = (12x^2 - 6x)e^{4x}
\end{aligned}
$$

and so

$$
\begin{aligned}
u_1' &= \frac{W_1}{W} = -12x^3 + 6x^2 \\
u_2' &= \frac{W_2}{W} = 12x^2 - 6x.
\end{aligned}
$$

Integration yields

$$\begin{aligned} u_1 &= \int (-12x^3 + 6x^2)\,dx = -3x^4 + 2x^3. \\ u_2 &= \int (12x^2 - 6x)\,dx = 4x^3 - 3x^2. \end{aligned}$$

Substitute into (57):

$$\begin{aligned} y_p &= (-3x^4 + 2x^3)e^{2x} + (4x^3 - 3x^2)xe^{2x} \\ &= (x^4 - x^3)e^{2x}. \end{aligned}$$

The general solution is thus

$$y = y_c + y_p = (c_1 + c_2 x - x^3 + x^4)e^{2x}. \tag{59}$$

Hence,

$$\begin{aligned} y' &= \left(c_2 - 3x^2 + 4x^3 + 2c_1 + 2c_2 x - 2x^3 + 2x^4\right) e^{2x} \\ &= \left(2c_1 + c_2 + 2c_2 x - 3x^2 + 2x^3 + 2x^4\right) e^{2x}. \end{aligned} \tag{60}$$

Substitute $y\,(0) = 1$ into (59) $\Rightarrow c_1 = 1$.
Substitute $y'\,(0) = 0$ into (60) $\Rightarrow 2c_1 + c_2 = 0;\quad c_2 = -2c_1 = -2$.
Solution to the initial-value problem:

$$y = (x^4 - x^3 - 2x + 1)e^{2x}.$$

12.

$$y'' - 4y = \frac{e^{2x}}{x}:$$

$$\begin{aligned} y'' - 4y &= 0 \\ &\Rightarrow m^2 - 4 = 0 \\ &\Rightarrow m = \pm 2 \end{aligned}$$

$$\begin{aligned} &\Rightarrow\ y_c(x) = c_1 e^{2x} + c_2 e^{-2x} \\ &\Rightarrow\ y_1 = e^{2x} \ \text{ and } \ y_2 = e^{-2x} \\ &\Rightarrow\ y_1' = 2e^{2x} \ \text{ and } \ y_2' = -2e^{-2x} \end{aligned}$$

$$\text{Hence, } W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{1}{x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{4x}}{x}$$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$$\text{and so } u_1' = \frac{W_1}{W} = \frac{1}{4x} \Rightarrow u_1 = \frac{\ln|x|}{4}$$

$$u_2' = \frac{W_2}{W} = -\frac{e^{4x}}{4x} \Rightarrow u_2 = -\int \frac{e^{4x}}{4x} dx$$

(The above integral is nonelementary and may be left in integral form. See Z&C 6th ed., sec 4.6, example 3.)

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \frac{e^{2x} \ln|x|}{4} - e^{-2x} \int \frac{e^{4x}}{4x} dx \end{aligned}$$

$$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{2x} \ln|x|}{4} - e^{-2x} \int \frac{e^{4x}}{4x} dx$$

$$\left[\text{Note: } \int \frac{e^{ax}}{x} dx \text{ does have a series solution:} \right.$$

$$\left. \int \frac{e^{ax}}{x} dx = \ln x + \frac{ax}{1 \cdot 1!} + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \cdots \right]$$

13.

$$y'' + 2y' + y = e^{-x} \ln x :$$

$$\begin{aligned} y'' + 2y' + y &= 0 \\ &\Rightarrow m^2 + 2m + 1 = 0 \\ &\Rightarrow m_1 = -1, m_2 = -1 \end{aligned}$$

$$\Rightarrow \quad y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$$
$$\Rightarrow \quad y_1 = e^{-x} \quad \text{and} \quad y_2 = x e^{-x}$$
$$\Rightarrow \quad y_1' = -e^{-x} \quad \text{and} \quad y_2' = e^{-x} - x e^{-x}$$

$$\begin{aligned}
\text{Hence,} \ W_1 &= \begin{vmatrix} 0 & x e^{-x} \\ e^{-x} \ln x & e^{-x} - x e^{-x} \end{vmatrix} = -x e^{-2x} \ln x \\
W_2 &= \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \ln x \end{vmatrix} = e^{-2x} \ln x \\
W &= \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x}
\end{aligned}$$

$$\begin{aligned}
\text{and so} \ u_1' &= \frac{W_1}{W} = -x \ln x \Rightarrow u_1 = -\frac{x^2 \ln x}{2} + \frac{x^2}{4} \\
u_2' &= \frac{W_2}{W} = \ln x \Rightarrow u_2 = x \ln x - x
\end{aligned}$$

$$\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \left(-\frac{x^2 \ln x}{2} + \frac{x^2}{4} \right) e^{-x} + (x \ln x - x)\, x e^{-x}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow y(x) &= c_1 e^{-x} + c_2 x e^{-x} - \frac{x^2 e^{-x} \ln x}{2} - \frac{3x^2 e^{-x}}{4} + x^2 e^{-x} \ln x \\
&= c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x} \left(\frac{\ln x}{2} - \frac{3}{4} \right).
\end{aligned}$$

14. First we solve for the complimentary solution in

$$y_c'' + y_c = 0$$

which has the auxiliary equation $m^2 + 1 = 0$ and solutions $m_1 = \mathbf{i}$ $m_2 = -\mathbf{i}$ so that

$$y_c = c_1 \cos x + c_2 \sin x.$$

For the particular solution we assume $y_p = u_1(x) \cos x + u_2(x) \sin x$ and using the technique of variation of parameters we obtain ($y_1 = \cos x$, $y_2 = \sin x$,

$f(x) = \cos^2 x)$

$$\begin{aligned} W &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1. \\ W_1 &= \begin{vmatrix} 0 & \sin x \\ \cos^2 x & \cos x \end{vmatrix} = -\sin x \cos^2 x. \\ W_2 &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \cos^2 x \end{vmatrix} = \cos^3 x = \cos x - \cos x \sin^2 x. \end{aligned}$$

$$\begin{aligned} u_1' &= \frac{W_1}{W} = -\sin x \cos^2 x. \\ u_1 &= -\int \sin x \cos^2 x \, dx = \frac{1}{3}\cos^3 x. \\ u_2' &= \cos x - \cos x \sin^2 x. \\ u_2 &= \int (\cos x - \cos x \sin^2 x) dx = \sin x - \frac{1}{3}\sin^3 x. \\ y_p &= u_1 y_1 + u_2 y_2 = \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x. \end{aligned}$$

Thus the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x.$$

15. The auxiliary equation is $m^2 + 3m + 2 = (m+2)(m+1) = 0$ with solutions $m_1 = -1$ and $m_2 = -2$ and so $y_c = c_1 e^{-x} + c_2 e^{-2x}$. For the particular solution $y_p = u_1 e^{-x} + u_2 e^{-2x}$ we find $(y_1 = e^{-x}$, $y_2 = e^{-2x}$, $f(x) = \sin e^x)$

$$\begin{aligned} W &= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}. \\ W_1 &= \begin{vmatrix} 0 & e^{-2x} \\ \sin e^x & -2e^{-2x} \end{vmatrix} = -e^{-2x} \sin e^x. \\ W_2 &= \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin e^x \end{vmatrix} = e^{-x} \sin e^x. \end{aligned}$$

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = e^x \sin e^x. \\
u_1 &= \int e^x \sin e^x dx = -\cos e^x. \\
u_2' &= \frac{W_1}{W} = -e^{2x} \sin e^x. \\
u_2 &= -\int e^{2x} \sin e^x dx \qquad (u = e^x, \quad du = e^x dx) \\
&= -\int u \sin u \, du = u \cos u - \int \cos u \, du = u \cos u - \sin u \\
&= e^x \cos e^x - \sin e^x.
\end{aligned}$$

Thus the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-x} \cos e^x + e^{-x} \cos e^x - e^{-2x} \sin e^x.$$

or

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x.$$

16. In standard form

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = \frac{1}{x^2} \sec(\ln x).$$

Thus we have, with $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$, $f(x) = \frac{1}{x^2} \sec(\ln x)$,

$$\begin{aligned}
W &= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}(\cos^2(\ln x) + \sin^2(\ln x)) = \frac{1}{x}. \\
W_1 &= \begin{vmatrix} 0 & \sin(\ln x) \\ \frac{1}{x^2} \sec(\ln x) & \frac{\cos(\ln x)}{x} \end{vmatrix} = -\frac{1}{x^2} \tan(\ln x). \\
W_2 &= \begin{vmatrix} \cos(\ln x) & 0 \\ -\frac{\sin(\ln x)}{x} & \frac{1}{x^2} \sec(\ln x) \end{vmatrix} = \frac{1}{x^2}.
\end{aligned}$$

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = -\frac{1}{x} \tan(\ln x). \\
u_1 &= -\int \frac{1}{x} \tan(\ln x) dx = \ln|\cos(\ln x)|. \\
u_2' &= \frac{W_2}{W} = \frac{1}{x}. \\
u_2 &= \int \frac{1}{x} dx = \ln x.
\end{aligned}$$

So the general solution is

$$\begin{aligned} y(x) &= c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2 \\ &= c_1 \cos(\ln x) + c_2 \sin(\ln x) + \cos(\ln x) \ln|\cos(\ln x)| + \ln(x) \sin(\ln x). \end{aligned}$$

17. In standard form we have

$$y'' - 2y' + 10y = 5 \sin x + \frac{e^x}{3} \tan 3x.$$

The auxiliary equation is $m^2 - 2m + 10 = 0$ with solutions $m_1 = 1 + 3\mathbf{i}$ and $m_2 = 1 - 3\mathbf{i}$ and consequently the complimentary solution is

$$y_c(x) = c_1 e^x \cos 3x + c_2 e^x \sin 3x.$$

We use $y_p = y_{p1} + y_{p2}$ where

$$y''_{p1} - 2y'_{p1} + 10y_{p1} = 5 \sin x$$

$$y''_{p2} - 2y'_{p2} + 10y_{p2} = \frac{e^x}{3} \tan 3x$$

and the superposition principle for non-homogeneous linear differential equations. We use undetermined coefficients to find $y_{p1} = A \cos x + B \sin x$ in

$$\begin{aligned} y''_{p1} - 2y'_{p1} + 10y_{p1} &= (-A \cos x - B \sin x) - 2(-A \sin x + B \cos x) + 10(A \cos x - B \sin x) \\ &= (9A - 2B) \cos x + (9B + 2A) \sin x = 5 \sin x \end{aligned}$$

and comparing coefficients for $\cos x$ and $\sin x$ yields $B = \frac{9}{2}A$ and $A = \frac{2}{17}$ and $B = \frac{9}{17}$ so that

$$y_{p1} = \frac{2}{17} \cos x + \frac{9}{17} \sin x.$$

For y_{p2} we use variation of parameters where $y_1 \equiv e^x \cos 3x$, $y_2 \equiv e^x \sin 3x$ and $f(x) \equiv \frac{e^x}{3} \tan 3x$ to obtain

$$\begin{aligned} W &= \begin{vmatrix} e^x \cos 3x & e^x \sin 3x \\ e^x (\cos 3x - 3 \sin 3x) & e^x (\sin 3x + 3 \cos 3x) \end{vmatrix} = 3e^{2x}. \\ W_1 &= \begin{vmatrix} 0 & e^x \sin 3x \\ \frac{e^x}{3} \tan 3x & e^x (\sin 3x + 3 \cos 3x) \end{vmatrix} = -\frac{e^{2x}}{3} \frac{\sin^2 3x}{\cos 3x} = -\frac{e^{2x}}{3} (\sec 3x - \cos 3x). \\ W_2 &= \begin{vmatrix} e^x \cos 3x & 0 \\ e^x (\cos 3x - 3 \sin 3x) & \frac{e^x}{3} \tan 3x \end{vmatrix} = \frac{e^{2x}}{3} \sin 3x. \end{aligned}$$

$$
\begin{aligned}
u_1' &= \frac{W_1}{W} = -\frac{1}{9}(\sec 3x - \cos 3x). \\
u_1 &= -\frac{1}{9}\int(\sec 3x - \cos 3x)dx = -\frac{1}{27}(\ln|\sec 3x + \tan 3x| - \sin 3x). \\
u_2' &= \frac{W_2}{W} = \frac{1}{9}\sin 3x. \\
u_2 &= \frac{1}{9}\int \sin 3x\, dx = -\frac{1}{27}\cos 3x. \\
y_{p2} &= u_1 y_1 + u_2 y_2 = -\frac{e^x}{27}(\ln|\sec 3x + \tan 3x|\cos 3x - \sin 3x\cos 3x) - \frac{e^x}{27}\cos 3x \sin 3x \\
&= -\frac{e^x}{27}\ln|\sec 3x + \tan 3x|\cos 3x.
\end{aligned}
$$

Thus we find the general solution

$$
\begin{aligned}
y(x) &= y_c + y_p = y_c + y_{p1} + y_{p2} \\
&= c_1 e^x \cos 3x + c_2 e^x \sin 3x + \frac{2}{17}\cos x + \frac{9}{17}\sin x - \frac{e^x}{27}\ln|\sec 3x + \tan 3x|\cos 3x.
\end{aligned}
$$

18. In standard form **on** $(0, \infty)$

$$
y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{1}{x^4}.
$$

Since $y_1 = x^2$ is a solution to the associated homogeneous equation we can generate a second linearly independent solution

$$
\begin{aligned}
y_2 &= x^2 \int \frac{e^{-\int \frac{1}{x}dx}}{x^4}dx \\
&= x^2 \int x^{-5}dx = -\frac{1}{4}x^{-2}
\end{aligned}
$$

it is simpler, however, to discard the constant and use $y_2 = x^{-2}$. Thus we have

$$
y_c = c_1 x^2 + c_2 x^{-2}.
$$

Using variation of parameters we have $y_p = u_1 y_1 + u_2 y_2$ and identifying

$f(x) \equiv x^{-4}$ we find

$$\begin{aligned} W &= \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -4x^{-1}. \\ W_1 &= \begin{vmatrix} 0 & x^{-2} \\ x^{-4} & -2x^{-3} \end{vmatrix} = -x^{-6}. \\ W_2 &= \begin{vmatrix} x^2 & 0 \\ 2x & x^{-4} \end{vmatrix} = x^{-2}. \end{aligned}$$

$$\begin{aligned} u_1' &= \frac{W_1}{W} = \frac{1}{4x^5}. \\ u_1 &= \int \frac{1}{4x^5} dx = -\frac{1}{16x^4}. \\ u_2 &= \frac{W_2}{W} = -\frac{1}{4x}. \\ u_2 &= -\int \frac{1}{4x} = -\frac{1}{4} \ln x. \\ y_p &= u_1 x^2 + u_2 x^{-2} = -\frac{1}{16x^2} - \frac{1}{4x^2} \ln x = -\frac{1}{4x^2}\left(\frac{1}{4} - \ln x\right). \end{aligned}$$

so that the general solution is

$$y(x) = c_1 x^2 + c_2 x^{-2} - \frac{1}{4x^2}\left(\frac{1}{4} - \ln x\right).$$