TWK2A
Undetermined coefficients (Section 4.4)
Solutions

1. First we solve the associated homogeneous differential equation with the assumption $y = e^{mx}$.

$$m^2 - 8m + 20 = 0, \qquad m_{1,2} = 4 \pm 2i$$

$$y_c = c_1 e^{4x}\sin(2x) + c_2 e^{4x}\cos(2x).$$

To find a particular solution we make the assumption

$$y_p = (Ax^2 + Bx + C) + (Dx + E)e^x$$

and apply the superposition principle.

$$\begin{aligned} 2A - 8(2Ax + B) + 20(Ax^2 + Bx + C) &= 100x^2 \\ 20A = 100 &\Rightarrow A = 5 \\ -16A + 20B = 0 &\Rightarrow B = 4 \\ 2A - 8B + 20C = 0 &\Rightarrow C = \frac{11}{10} \end{aligned}$$

$$((Dx+E)e^x + 2De^x) - 8((Dx+E)e^x + De^x) + 20(Dx+E)e^x = -26xe^x$$

$$\begin{aligned} D - 8D + 20D = -26 &\Rightarrow D = -2 \\ E + 2D - 8E - 8D + 20E = 0 &\Rightarrow E = -\frac{12}{13} \end{aligned}$$

Thus the general solution is

$$y = c_1 e^{4x}\sin(2x) + c_2 e^{4x}\cos(2x) + 5x^2 + 4x + \frac{11}{10} - \left(2x + \frac{12}{13}\right)e^x.$$

2. First we solve the associated homogeneous differential equation with the assumption $y = e^{mx}$.

$$m^2 + 3 = 0, \qquad m_{1,2} = \pm\sqrt{2}i$$

$$y_c = c_1 \sin(\sqrt{3}x) + c_2 \cos(\sqrt{3}x).$$

To find a particular solution we make the assumption

$$y_p = (Ax^2 + Bx + C)e^{3x}.$$

$$9(Ax^2+Bx+C)e^{3x}+6(2Ax+B)e^{3x}+4Ae^{3x}+3(Ax^2+Bx+C)e^{3x} = -48x^2e^{3x}$$

$$\begin{aligned} 9A + 3A = -48 \;&\Rightarrow\; A = -4 \\ 9B + 12A + 3B = 0 \;&\Rightarrow\; B = 4 \\ 9C + 6B + 4A + 3C = 0 \;&\Rightarrow\; C = -\frac{4}{3} \end{aligned}$$

Thus the general solution is

$$y = c_1 \sin(\sqrt{3}x) + c_2 \cos(\sqrt{3}x) - \left(4x^2 - 4x + \frac{4}{3}\right) e^{3x}.$$

3. First we solve the associated homogeneous differential equation with the assumption $y = e^{mx}$.

$$4m^2 - 4m - 3 = (2m-3)(2m+1) = 0, \qquad m_1 = \frac{3}{2}, \quad m_2 = -\frac{1}{2}$$

$$y_c = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}.$$

To find a particular solution we make the assumption

$$y_p = A\cos(2x) + B\sin(2x).$$

$$\begin{gathered} -16(A\cos(2x) + B\sin(2x)) - 8(-A\sin(2x) + B\cos(2x)) \\ -3(A\cos(2x) + B\sin(2x)) = \cos(2x) \end{gathered}$$

$$\begin{aligned} -16B + 8A - 3B = 0 \;&\Rightarrow\; \frac{8}{19}A = B \\ -16A - 8B - 3A = 1 \;&\Rightarrow\; -19A - \frac{64}{19}A = 1 \\ A = -\frac{19}{425}, \qquad &B = -\frac{8}{425} \end{aligned}$$

$$y = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x} - \frac{1}{425}(19\cos(2x) - 8\sin(2x)).$$

4. First we solve the associated homogeneous differential equation with the assumption $y = e^{mx}$.

$$m^2 + 1 = 0, \qquad m_{1,2} = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x.$$

To find a particular solution we make the assumption (multiply by x since $\sin x$ is a solution to the associated homogeneous equation)

$$y_p = (Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x.$$

$$\begin{aligned}
&-(Ax^2 + Bx)\sin x + 2(2Ax + B)\cos x + 2A\sin x \\
&-(Cx^2 + Dx)\cos x - 2(2Cx + D)\sin x + 2C\cos x \\
&+(Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x = 2x\sin x
\end{aligned}$$

$$\begin{aligned}
4A &= 0 \\
-4C &= 2 \\
2B + 2C &= 0 \\
-2D + 2A &= 0
\end{aligned}$$

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}x\sin x - \frac{1}{2}x^2\cos x.$$

5. First we solve the associated homogeneous differential equation with the assumption $y = e^{mx}$.

$$m^2 - 1 = 0, \qquad m_{1,2} = \pm 1$$

$$y_c = c_1 e^x + c_2 e^{-x}.$$

To find a particular solution we make the assumption (multiply by x since $\cosh x \equiv \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ is a solution to the associated homogeneous equation).

$$y_p = Ax\cosh x + Bx\sinh x.$$

$$\begin{aligned}
Ax\cosh x + 2A\sinh x + Bx\sinh x + 2B\cosh x \\
-Ax\cosh x - Bx\sinh x = \cosh x \\
A = 0, \quad 2B = 1
\end{aligned}$$

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x \sinh x.$$

$$\begin{aligned} y(0) = 2 \quad &\Rightarrow \quad c_1 + c_2 = 2 \\ y'(0) = 12 \quad &\Rightarrow \quad c_1 - c_2 = 12 \end{aligned}$$

Thus the solution to the initial value problem is

$$y = 7e^x - 5e^{-x} + \frac{1}{2} x \sinh x.$$

6.

$$y'' + y' - 6y = 2x \tag{1}$$

The auxiliary equation for the associated homogeneous equation is

$$\begin{aligned} m^2 + m - 6 & \quad 0 \\ (m+3)(m-2) &= 0 \end{aligned}$$

The roots are $m_1 = -3$ and $m_2 = 2$ and the complementary function is therefore given by

$$y_c = c_1 e^{-3x} + c_2 e^{2x}. \tag{2}$$

A particular solution will have the form

$$y_p = Ax + B. \tag{3}$$

Therefore

$$y_p' = A \tag{4}$$

and

$$y_p'' = 0. \tag{5}$$

Substitute (3) - (5) in (1)

$$\begin{aligned} y_p'' + y_p' - 6y_p = 0 + A - 6(Ax + B) &= 2x \\ -6Ax + (A - 6B) &= 2x \end{aligned}$$

Equating the coefficients of x on either side of the equation we find

$$\begin{aligned} -6A = 2; & \qquad A = -\frac{1}{3} \\ A - 6B = 0; & \qquad B = \frac{A}{6} = -18 \end{aligned}$$

Equation (3) now becomes

$$y_p = -\frac{1}{3}x - \frac{1}{18}. \tag{6}$$

The general solution follows from (2) and (6):

$$y = y_c + y_p = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{3}x - \frac{1}{18}$$

7.

$$y'' + 4y = (x^2 - 3)\sin 2x \tag{7}$$

The auxiliary equation for the associated homogeneous equation is

$$m^2 + 4 = 0 \Rightarrow m^2 = -4$$

The roots are $m_1 = 2\mathbf{i}$; $m_2 = -2\mathbf{i}$ and so the complementary function is given by

$$y_c = c_1 \cos 2x + c_2 \sin 2x. \tag{8}$$

In order to eliminate duplication of terms in y_c, a particular solution must have the form

$$y_p = (Ax^3 + Bx^2 + Cx)\sin 2x + (Dx^3 + Ex^2 + Fx)\cos 2x \tag{9}$$

Hence,

$$\begin{aligned} y_p' &= [-2Dx^3 + (3A - 2E)x^2 + 2(B - F)x + C]\sin 2x \\ &\quad +[2Ax^3 + (2B + 3D)x^2 + 2(C + E)x + F]\cos 2x \end{aligned} \tag{10}$$

and

$$\begin{aligned} y_p'' &= [-4Ax^3 - 4(3D + B)x^2 + 2(3A - 2C - 4E)x + 2(B - 2F)]\sin 2x \\ &+ [-4Dx^3 + 4(3A - E)x^2 + 2(4B + 3D - 2F)x + 2(2C + E)]\cos 2x \end{aligned} \tag{11}$$

Substitute (9) and (11) in (7) to obtain

$$\begin{aligned} y_p'' + 4y_p &= [-12Dx^2 + (6A - 8E)x + (2B - 4F)]\sin 2x \\ &+ [12Ax^2 + (8B + 6D)x + (4C + 2E)]\cos 2x \\ &= x^2 \sin 2x - 3\sin 2x \end{aligned}$$

Equating the coefficients of similar linearly independent functions on both sides of the equation, we find

$$\begin{array}{ll} -12D = 1 & 6A - 8E = 0 \\ 2B - 4F = -3 & 12A = 0 \\ 8B + 6D = 0 & 4C + 2E = 0 \end{array}$$

These equations yield

$$A = 0; \quad B = \frac{1}{16}; \quad C = 0; \quad D = -\frac{1}{12}; \quad E = 0; \quad F = \frac{25}{32}.$$

Hence, a particular solution is given by

$$y_p = \frac{1}{16}x^2 \sin 2x + \left(-\frac{1}{12}x^3 + \frac{25}{32}x\right) \cos 2x. \tag{12}$$

The general solution follows from (8) and (12):

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{16}x^2 \sin 2x + \left(-\frac{1}{12}x^3 + \frac{25}{32}x\right) \cos 2x. \end{aligned}$$

8.

$$y^{(4)} - y'' = 4x + 2xe^{-x} \tag{13}$$

The auxiliary equation for the associated homogeneous equation is

$$m^4 - m^2 = 0 \Rightarrow m^2(m^2 - 1) = 0$$

The roots are $m_1 = m_2 = 0$, $m_3 = 1$ and $m_4 = -1$ and the complementary function is therefore given by

$$y_c = c_1 + c_2x + c_3e^x + c_4e^{-x}. \tag{14}$$

In order to eliminate duplication of terms in y_c, a particular solution must have the form

$$y_p = Ax^3 + Bx^2 + (Dx^2 + Ex)e^{-x}. \tag{15}$$

The derivatives of y_p are:

$$\begin{aligned} y_p' &= 3Ax^2 + 2Bx + [-Dx^2 + (2D - E)x + E]e^{-x} \\ y_p'' &= 6Ax + 2B + [Dx^2 + (-4D + E)x + 2(D - E)]e^{-x} \qquad (16) \\ y_p''' &= 6A + [-Dx^2 + (6D - E)x + 3(E - 2D)]e^{-x} \\ y_p^{(4)} &= [Dx^2 + (E - 8D)x + 4(3D - E)]e^{-x} \qquad (17) \end{aligned}$$

We substitute (16) and (17) in (14):

$$y_p^{(4)} - y_p'' = -6Ax - 2B + [-4Dx + (10D - 2E)]e^{-x} = 4x + 2xe^{-x}$$

Equating the coefficients of similar linear independent functions on both sides of the equation, we find

$$\begin{aligned} -6A = 4 \Rightarrow &\quad A = -\frac{3}{3} \\ -2B = 0 \Rightarrow &\quad B = 0 \\ -4D = 2 \Rightarrow &\quad D = -\frac{1}{2} \\ 10D - 2E = 0 \Rightarrow &\quad E = 5D = -\frac{5}{2} \end{aligned}$$

Equation (15) becomes

$$y_p = -\frac{2}{3}x^3 - \frac{1}{2}(x^2 + 5x)e^{-x}. \tag{18}$$

The general solution follows from (14) and (18):

$$y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - \frac{2}{3}x^3 - \frac{1}{2}(x^2 + 5x)e^{-x}.$$

9.

$$4y'' + 9y = 15 \tag{19}$$

Auxiliary equation for complementary function:

$$4m^2 + 9 = 0 \Rightarrow m = \pm\frac{3}{2}\mathbf{i}$$

Therefore

$$y_c = c_1 \cos\frac{3}{2}x + c_2 \sin\frac{3}{2}x$$

For the particular solution we assume the form

$$y = A$$

Then

$$y' = 0; \quad y'' = 0$$

Substitute into (19):

$$0 + 9A = 15 \Rightarrow A = \frac{5}{3}$$

Therefore

$$y_p = \frac{5}{3}$$

The general solution is given by

$$y = y_c + y_p = c_1 \cos \frac{3}{2}x + c_2 \sin \frac{3}{2}x + \frac{5}{3}$$

10.

$$y'' + 2y' = 2x + 5 - e^{-2x} \tag{20}$$

Auxiliary equation:

$$m^2 + 2m = 0 \Rightarrow m(m+2) = 0 \Rightarrow m = 0 \text{ or } \quad -2$$

Hence

$$y_c = c_1 + c_2 e^{-2x}.$$

Since a constant term as well as the function e^{-2x} already occurs in the complementary function, the particular solution must have the form

$$\begin{aligned} y &= (Ax + B + Ce^{-2x})x \\ &= Ax^2 + Bx + Cxe^{-2x} \end{aligned}$$

Then

$$y' = 2Ax + B + C(1 - 2x)e^{-2x}$$

and

$$\begin{aligned} y'' &= 2A + C[-2 - 2(1 - 2x)]e^{-2x} \\ &= 2A + 4C(x - 1)e^{-2x} \end{aligned}$$

We substitute the solution in (20):

$$\begin{aligned} 2A + 4C(x-1)e^{-2x} + 4Ax + 2B + 2C(1-2x)e^{-2x} &= 2x + 5 - e^{-2x} \\ \Rightarrow 2(A + B) + 4Ax - 2Ce^{-2x} &= 2x + 5 - e^{-2x} \end{aligned}$$

Compare coefficients:

$$\begin{array}{lr} x: & 4A = 2 \Rightarrow A = \frac{1}{2} \\ 1\,(=x^0): & 2(A+B) = 5 \Rightarrow B = \frac{5}{2} - A = 2 \\ e^{-2x}: & -2C = -1 \Rightarrow C = \frac{1}{2} \end{array}$$

Hence

$$y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}.$$

The general solution is given by

$$y = y_c + y_p = c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}.$$

11.

$$y'' + 2y' - 24y = 16 - (x+2)e^{4x} \tag{21}$$

Auxiliary equation:

$$m^2 + 2m - 24 = 0 \Rightarrow (m+6)(m-4) = 0 \Rightarrow m = 4 \text{ or } -6$$

Hence

$$y_c = c_1e^{4x} + c_2e^{-6x}$$

Since e^{4x} already occurs in y_c, the particular solution must have the form

$$\begin{aligned} y &= A + x(Bx + C)e^{4x} \\ &= A = (Bx^2 + Cx)e^{4x} \end{aligned}$$

Then

$$\begin{aligned} y' &= \left[(2Bx + C) + 4(Bx^2 + (x)\right] e^{4x} \\ &= \left[4Bx^2 + (2B + 4C)x + C\right] e^{4x} \end{aligned}$$

and

$$\begin{aligned} y'' &= \left[7Bx + (2B + 4C) + 4\{4Bx^2 + (2B + 4C)x + C\}\right] e^{4x} \\ &= \left[16Bx^2 + 16(B + C)x + (2B + 8C)\right] e^{4x} \end{aligned}$$

Substitute the particular solution in (21):

$$\begin{aligned} \left[16Bx^2 + 216(B + C)x + (2B + 8C)\right] e^{4x} &+ \left[8Bx^2 + (4B + 8C)x + 2C\right] e^{4x} \\ -24A - (24Bx^2 + 24Cx)e^{4x} &= 16 - (x + 2)e^{4x} \\ \Rightarrow -24A + 20Bxe^{4x} + (2B + 10C)e^{4x} &= 16 - xe^{4x} - 2e^{4x} \end{aligned}$$

Compare coefficients:

$$\begin{array}{ll} 1\,(=x^0): & -24A = 16 \Rightarrow A = -\frac{2}{3} \\ xe^{4x}: & 20B = -1 \Rightarrow B = -\frac{1}{20} \\ e^{4x}: & 2B + 10C = -2 \Rightarrow C = \frac{1}{10}(-2B-2) = -\frac{19}{100} \end{array}$$

The particular solution is therefore given by

$$y_p = -\frac{2}{3} - \left(\frac{1}{20}x^2 + \frac{19}{100}x\right) e^{4x}$$

and the general solution is

$$y = y_c + y_p = \left(c_1 - \frac{1}{20}x^2 - \frac{19}{100}x\right) e^{4x} + c_2 e^{-6x} - \frac{2}{3}.$$

12.

$$2y'' + 3y' - 2y = 14x^2 - 4x - 11 \tag{22}$$

with initial condition

$$y(0) = 0; \qquad y'(0) = 0 \tag{23}$$

Auxiliary equation:

$$2m^2 + 3m - 2 = 0 \Rightarrow (2m-1)(m+2) = 0 \Rightarrow m = \frac{1}{2} \text{ or } -2$$

Complementary function:

$$y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} \tag{24}$$

The particular solution has the form

$$y = Ax^2 + Bx + c$$

Then

$$y' = 2Ax + B$$

and

$$y'' = 2A$$

Substitute in (22):

$$\begin{aligned} 4A + 3(2Ax + B) - 2(Ax^2 + Bx + C) &= 14x^2 - 4x - 11 \\ -2Ax^2 + (6A - 2B)x^2 + (4A + 3B - 2C) &= 14x^2 - 4x - 11 \end{aligned}$$

Compare coefficients:

$$\begin{array}{lr} x^2: & -2A = 14 \Rightarrow A = -7 \\ x: & 6A - 2B = -4 \Rightarrow B = \frac{1}{2}(6A+4) = -19 \\ 1\,(=x^0): & 4A + 3B - 2C = -11 \Rightarrow C = \frac{1}{2}(4A+3B+11) = -37 \end{array}$$

Particular solution:

$$y_p = -(7x^2 + 19x + 37).$$

The general solution is

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} - (7x^2 + 19x + 37). \tag{25}$$

Therefore

$$y' = \frac{1}{2} c_1 e^{\frac{1}{2}x} - 2c_2 e^{-2x} - (14x + 19). \tag{26}$$

Substitute (23) in (25) and (26):

$$\begin{aligned} y\,(0) &= 0 \Rightarrow c_1 + c_2 - 37 = 0 \\ y'\,(0) &= 0 \Rightarrow \frac{1}{2} c_1 - 2c_2 - 19 = 0 \end{aligned}$$

Solving the equations, we find

$$c_1 = \frac{186}{5} \quad ; \quad c_2 = -\frac{1}{5}$$

The solution to the initial-value problem:

$$y = \frac{186}{5} e^{\frac{1}{2}x} - \frac{1}{5} e^{-2x} - (7x^2 + 19x + 37).$$

13.

$$y''' - y' = x^3 :$$

$$\begin{aligned} y''' - y' = 0 &\Rightarrow m^3 - m = 0 \Rightarrow m\left(m^2 - 1\right) \\ &\Rightarrow m_1 = 0, m_2 = 1, m_3 = -1 \\ &\Rightarrow y_c(x) = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-x} \\ &= c_1 + c_2 e^{x} + c_3 e^{-x} \end{aligned}$$

For y_p :

$g:$	x^3	$\Rightarrow$	Ax^3
$g':$	$3x^2$	$\Rightarrow$	Bx^2
$g'':$	$6x$	$\Rightarrow$	Cx
$g''':$	6	$\Rightarrow$	$\begin{cases} D \text{ (duplicated in } y_c) \\ \downarrow \\ Dx \text{ (duplicated above)} \\ \downarrow \\ Dx^2 \text{ (duplicated above)} \\ \downarrow \\ Dx^3 \text{ (duplicated above)} \\ \downarrow \\ Dx^4 \end{cases}$
$g^{(iv)}:$	0 (not new)		

$$\Rightarrow \quad y_p = Ax^3 + Bx^2 + Cx + Dx^4$$
$$\Rightarrow \quad y_p' = 4Dx^3 + 3Ax^2 + 2Bx + C$$
$$\Rightarrow \quad y_p''' = 24Dx + 6A$$

$$\text{Hence, } \; y_p''' - y_p' = x^3$$

$$\Rightarrow \quad -4Dx^3 - 3Ax^2 + (24D - 2B)\,x + (6A - C) = x^3$$
$$\Rightarrow \quad A = 0, B = -3, C = 0, D = -\frac{1}{4}$$

$$\text{So } \; y\,(x) = c_1 + c_2 e^x + c_3 e^{-x} - 3x^2 - \frac{1}{4}x^4.$$

14.

$$y''' - 6y'' = 3 - \cos x :$$

$$y''' - 6y'' = 0 \Rightarrow m^3 - 6m^2 = 0 \Rightarrow m^2\,(m - 6)$$
$$\Rightarrow m_1 = 0, m_2 = 0, m_3 = 6$$
$$\Rightarrow y_c(x) = c_1 + c_2 x + c_3 e^{6x}$$

For y_p :

$g:$	$3-\cos x$	$\Rightarrow$	$A\cos x,\ \underbrace{B}_{\text{duplicated in } y_c} \longrightarrow \underbrace{Bx}_{\text{duplicated in } y_c} \longrightarrow Bx^2$
$g':$	$\sin x$	$\Rightarrow$	$C\sin x$
$g'':$	$\cos x$ (not new)		

$$\begin{aligned}
&\Rightarrow\ y_p = A\cos x + Bx^2 + C\sin x\\
&\Rightarrow\ y_p'' = -A\cos x - C\sin x + 2B\\
&\Rightarrow\ y_p''' = A\sin x - C\cos x
\end{aligned}$$

$$\text{Hence, } \ y_p''' - 6y_p'' = 3 - \cos x$$

$$\begin{aligned}
&\Rightarrow\ (A+6C)\sin x + (6A-C)\cos x - 12B = 3 - \cos x\\
&\Rightarrow\ A = -\frac{6}{37}, B = -\frac{1}{4}, C = \frac{1}{37}
\end{aligned}$$

$$\text{So } \ y(x) = c_1 + c_2 x + c_3 e^{6x} - \frac{6}{37}\cos x - \frac{1}{4}x^2 + \frac{1}{37}\sin x.$$

15.

$$y^{(4)} + 3y''' + 3y'' - 7y' = (x-1)^2 :$$

$$y^{(4)} + 3y''' + 3y'' - 7y' = 0$$

$$\begin{aligned}
&\Rightarrow\ m^4 + 3m^3 + 3m^2 - 7m = 0\\
&\Rightarrow\ m\left(m^3 + 3m^2 + 3m - 7\right) = 0\\
&\Rightarrow\ m(m-1)\left(m^2 + 4m + 7\right) = 0\\
&\Rightarrow\ m_1 = 0, m_2 = 1, m_3 = -2 + \sqrt{3}\mathbf{i}, m_4 = -2 - \sqrt{3}\mathbf{i}
\end{aligned}$$

$$\Rightarrow y_c(x) = c_1 + c_2 e^x + c_3 e^{-2x}\cos\left(\sqrt{3}x\right) + c_4 e^{-2x}\sin\left(\sqrt{3}x\right)$$

For y_p :

$g:$	$x^2 - 2x + 1$	$\Rightarrow$	$\left\{ Ax^2, Bx, \underbrace{C}_{\text{duplicated in } y_c} \downarrow \underbrace{Cx}_{\text{duplicated by } Bx} \downarrow \underbrace{Bx^2}_{\text{duplicated by } Ax^2} \downarrow Cx^3 \right.$
$g':$	$2x - 2$ (not new)		

$$\begin{aligned}
&\Rightarrow \quad y_p = Ax^2 + Bx + Cx^3 \\
&\Rightarrow \quad y_p' = 2Ax + B + 3Cx^2 \\
&\Rightarrow \quad y_p'' = 2A + 6Cx \\
&\Rightarrow \quad y_p''' = 6C \\
&\Rightarrow \quad y_p^{(iv)} = 0
\end{aligned}$$

$$\text{Hence,} \quad y^{(4)} + 3y''' + 3y'' - 7y' = (x-1)^2$$

$$\begin{aligned}
&\Rightarrow \quad -21Cx^2 + (18C - 14A)\,x + (-7B + 6A + 18C) = x^2 - 2x + 1 \\
&\Rightarrow \quad A = \frac{4}{49}, B = -\frac{67}{343}, C = -\frac{1}{21}
\end{aligned}$$

$$\begin{aligned}
\text{So} \quad y(x) \quad &= \quad c_1 + c_2 e^{x} + c_3 e^{-2x}\cos\left(\sqrt{3}x\right) + c_4 e^{-2x}\sin\left(\sqrt{3}x\right) + \\
& \quad \frac{4}{49}x^2 - \frac{67}{343}x - \frac{1}{21}x^3.
\end{aligned}$$