TWK2A
Homogeneous linear DEs with constant coefficients (Section 4.3)
Solutions

1.

$$\begin{aligned} & y'' - 4y' + 5y = 0 \\ \text{assume} \quad & y = e^{mx} \\ \Rightarrow \quad & m^2 e^{mx} - 4me^{mx} + 5e^{mx} = 0 \\ \Rightarrow \quad & m^2 - 4m + 5 = 0 \\ \Rightarrow \quad & m_1 = 2 + i \quad m_2 = 2 - i \end{aligned}$$

So $\alpha = 2, \quad \beta = 1$. Hence,

$$y = c_1 e^{2x} \cos(x) + c_2 e^{2x} \sin(x)$$

2.

$$y^{(4)} - 2y'' + y = 0$$

Assume $y = e^{mx}$:

$$\begin{aligned} \Rightarrow \quad & m^4 e^{mx} - 2m^2 e^{mx} + e^{mx} = 0 \\ \Rightarrow \quad & m^4 - 2m^2 + 1 = 0 \end{aligned}$$

Define $p = m^2$:

$$\begin{aligned} & p^2 - 2p + 1 = 0 \\ \Rightarrow \quad & (p-1)(p-1) = 0 \\ \Rightarrow \quad & p_1 = 1, \quad p_2 = 1 \quad \text{(two identical solutions)} \end{aligned}$$

Now $m = \sqrt{p}$. So we have

$$\begin{aligned} m_1 &= 1 \\ m_2 &= -1 \\ m_3 &= 1 \\ m_4 &= -1 \end{aligned}$$

Solutions of DE are then

$$\begin{aligned} e^{m_1 x} &= e^x \\ e^{m_2 x} &= e^{-x} \\ e^{m_3 x} &= e^x \longrightarrow xe^x \\ e^{m_4 x} &= e^{-x} \longrightarrow xe^{-x} \end{aligned}$$

and so

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 x e^x + c_4 x e^{-x}.$$

3.

$$16y^{(4)} + 24y'' + 9y = 0$$

Assume $y = e^{mx}$:

$$16m^x + 24m^2 + 0 = 0$$

Define $p = m^2$:

$$\begin{aligned} &\Rightarrow \quad 16p^2 + 24p + 9 = 0 \\ &\Rightarrow \quad p_1 = -\frac{3}{4} \qquad p_2 = -\frac{3}{4} \\ &\Rightarrow \quad m_1 = \sqrt{-\frac{3}{4}} = \sqrt{\frac{3}{4}}i \\ & \quad m_2 = -\sqrt{\frac{3}{4}}i \\ & \quad m_3 = \sqrt{\frac{3}{4}}i \\ & \quad m_4 = -\sqrt{\frac{3}{4}}i \end{aligned}$$

Hence $\alpha = 0, \quad \beta = \sqrt{\frac{3}{4}}$.
So

$$\begin{aligned} y(x) &= c_1 e^0 \cos\left(\sqrt{\frac{3}{4}}x\right) + c_2 e^0 \sin\left(-\sqrt{\frac{3}{4}}x\right) \\ &+ c_3 e^0 \cos\left(\sqrt{\frac{3}{4}}x\right) + c_4 e^0 \sin\left(-\sqrt{\frac{3}{4}}x\right) \end{aligned}$$

and

$$y(x) = c_1 \cos\left(\sqrt{\frac{3}{4}}x\right) + c_2 \sin\left(\sqrt{\frac{3}{4}}x\right) + c_3 x \cos\left(\sqrt{\frac{3}{4}}x\right) + c_4 x \sin\left(\sqrt{\frac{3}{4}}x\right).$$

4.

$$y'' + y' + 2y = 0 \qquad y(0) = y'(0) = 0$$

Assume $y = e^{mx}$:

$$\begin{aligned} \text{So } \ m^2 + m + 2 = 0 \Rightarrow m &= \frac{-1 \pm \sqrt{1 - 4(1)(2)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1}{2} + \frac{\sqrt{7}}{2}i \quad \text{or} \quad \frac{-1}{2} - \frac{\sqrt{7}}{2}i \end{aligned}$$

So $\alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{7}}{2} = \sqrt{\frac{7}{4}}$.

$$\begin{aligned} y(x) &= c_1 e^{-\frac{1}{2}x} \cos\left(\sqrt{\frac{7}{4}}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\sqrt{\frac{7}{4}}x\right) \\ y'(x) &= c_1 e^{-\frac{1}{2}x}\left(-\sin\left(\sqrt{\frac{7}{4}}x\right)\right) + c_1\left(-\frac{1}{2}\right) e^{-\frac{1}{2}x} \cos\left(\sqrt{\frac{7}{4}}x\right) \\ &+ c_2 e^{-\frac{1}{2}x} \cos\left(\sqrt{\frac{7}{4}}x\right) + c_2\left(-\frac{1}{2}\right) e^{-\frac{1}{2}x} \sin\left(\sqrt{\frac{7}{4}}x\right) \end{aligned}$$

$$\begin{aligned} y(0) &= c_1 + 0 = 0 \quad \Rightarrow c_1 = 0 \\ y'(0) &= 0 - \frac{1}{2}c_1 + c_2 + 0 = c_2 - \frac{1}{2}c_1 \Rightarrow c_2 = 0 \end{aligned}$$

So $y(x) = 0$ (Check that this **is** a solution to the IVP)
Verify that this solution makes sense in the light of Theorem 4.1.

5.

$$y'' - 2y' + 2y = 0 \qquad y(0) = 1 \qquad y\left(\frac{\pi}{2}\right) = 1$$

Assume $y = e^{mx}$:

$$\begin{aligned} & m^2 - 2m + 2 = 0 \\ \Rightarrow \quad & m_1 = 1 + i \qquad m_2 = 1 - i \end{aligned}$$

So $\alpha = 1, \quad \beta = 1.$

$$y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x)$$

$$\begin{aligned} y(0) &= c_1 = 1 \\ y\left(\frac{\pi}{2}\right) &= c_1 e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right) + c_2 e^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}\right) \\ &= c_2 e^{\pi} = 1 \\ \Rightarrow c_2 &= e^{-\frac{\pi}{2}} \\ y(x) &= e^x \cos(x) + e^{-\pi} e^x \sin(x) \end{aligned}$$

6. Cubic auxiliary equation:

$am^3 + 6m^2 + cm + d = 0$ comes from $ay''' + by'' + cy' + dy = 0$ with $y = e^{mx}$

and so

$$m_1 = -\frac{1}{2} \qquad m_2 = 3 + i$$

Since complex roots always occur in conjugate pairs

$$m_3 = 3 - i$$

So

$$\begin{aligned} am^3 + bm^2 + cm + d &= \left(m + \frac{1}{2}\right)(m - 3 - i)(m - 3 + i) \\ &= \left(m + \frac{1}{2}\right)\left(m^2 - 3m + mi - 3m + 9 - 3i - mi + 3i - i^2\right) \\ &= \left(m + \frac{1}{2}\right)\left(m^2 - 6m + 10\right) \\ &= m^3 - \frac{11}{2}m^2 + 7m + 5 \end{aligned}$$

So $a = 1, \quad b = -\frac{11}{2}, \quad c = 7, \quad d = 5$
The DE is thus

$$y''' - \frac{11}{2}y'' + 7y' + 5y = 0.$$

7.

$$y'' - 3y' + 2y = 0$$

Auxiliary equation:

$$\begin{aligned} m^2 - 3m' + 2m &= 0 \\ (m-1)(m-2) &= 0 \\ m = 1 \text{ or } 2 & \end{aligned}$$

Solution:

$$y = c_1 e^x + c_2 e^{2x}.$$

8.

$$y'' - 10y' + 25y = 0$$

Auxiliary equation:

$$\begin{aligned} m^2 - 5m + 25 &= 0 \\ (m-5)^2 &= 0 \\ m &= 5 \text{ (twice)} \end{aligned}$$

Solution:

$$y = c_1 e^{5x} + c_2 x e^{5x}.$$

9.

$$2y'' - 3y' + 4y = 0$$

Auxiliary equation:

$$\begin{aligned} 2m^2 - 3m + 4 &= 0 \\ m = \frac{3 \pm \sqrt{-23}}{4} &= \frac{3}{4} \pm \frac{\sqrt{23}}{4} i \end{aligned}$$

Solution:

$$y = e^{\frac{3}{4}x}\left[c_1 \cos\left(\frac{\sqrt{23}}{4}\,x\right) + c_2 \sin\left(\frac{\sqrt{23}}{4}\,x\right)\right].$$

10.

$$y''' - 6y'' + 12y' - 8y = 0$$

Auxiliary equation:

$$f(m) := m^3 - 6m^2 + 12m - 8 = 0$$

By inspection we find $f(2) = 0$, which means that $(m - 2)$ is a factor. Division gives

$$f(m) = (m - 2)(m^2 - 4m + 2) = 0.$$

and so

$$(m - 2)^3 = 0$$

Hence,

$$m = 2 \quad \text{(three times)}$$

and the solution is given by

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}.$$

11.

$$2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2} = 0$$

Auxiliary equation:

$$\begin{aligned} 2m^5 - 7m^4 + 12m^3 + 8m^2 &= 0 \\ m^2(2m^3 - 7m^2 + 12m + 8) &= 0 \end{aligned}$$

By inspection we establish that putting $m = -\frac{1}{2}$ reduces the expression in parentheses to zero, which means that $\left(m + \frac{1}{2}\right)$ [and therefore $(2m + 1)$] is a factor of the expression. By division we obtain

$$m^2(2m + 1)(m^2 - 4m + 8) = 0.$$

Therefore, the five roots are

$$m = 0 \text{ (twice)}, \ -\frac{1}{2} \quad \text{or} \quad 2 \pm 2i$$

Solution:

$$y = c_1 + c_2 s + c_3 e^{-\frac{1}{2}s} + e^{2s}(c_4 \cos 2s + c_5 \sin 2s)$$

12.

$$4y'' - 4y' - 3y = 0$$

Auxiliary equation:

$$\begin{aligned} 4m^2 - 4m - 3 &= 0 \\ (2m-3)(2m+1) &= 0 \\ m = \frac{3}{2} \quad \text{or} \quad -\frac{1}{2} \end{aligned}$$

Hence, the solution is given by

$$y = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}$$

so that

$$y' = \frac{3}{2} c_1 e^{\frac{3}{2}x} - \frac{1}{2} c_2 e^{-\frac{1}{2}x}.$$

Substituting the initial-values:

$$\begin{aligned} x = 0; y = 1 &\Rightarrow c_1 + c_2 = 1 \\ x = 0; y' = 5 &\Rightarrow \tfrac{3}{2} c_1 - \tfrac{1}{2} c_2 = 5 \end{aligned}$$

The solution of these two linear equations gives $c_1 = \frac{11}{4}$; $c_2 = -\frac{7}{4}$, and so the solution of the IVP is

$$y(x) = \frac{1}{4}\left(11\, e^{\frac{3}{2}x} - 7\, e^{-\frac{1}{2}x}\right).$$

13.

$$y'' + 4y = 0$$

Auxiliary equation:

$$\begin{aligned} m^2 + 4 &= 0 \\ \Rightarrow m &= \pm 2i \\ \Rightarrow y(x) &= c_1 \cos 2x + c_2 \sin 2x \end{aligned}$$

Substitute the boundary values:

$$\begin{aligned} y(0) = 0 &\Rightarrow c_1 = 0 \\ y\left(\tfrac{\pi}{4}\right) = 1 &\Rightarrow c_2 = 1 \end{aligned}$$

The solution therefore is

$$y = \sin 2x.$$

14.

$$y'' + 4y' - y = 0$$

Auxiliary equation:

$$\begin{aligned} m^2 + 4m - 1 &= 0 \\ \Rightarrow m &= \frac{-4 \pm \sqrt{20}}{2} = -2 \pm \sqrt{5} \end{aligned}$$

General solution:

$$y = c_1 e^{\left(-2+\sqrt{5}\right)x} + c_2 e^{\left(-2-\sqrt{5}\right)x}.$$

15.

$$3y'' + y = 0$$

Auxiliary equation:

$$\begin{aligned} 3m^2 + 1 &= 0 \\ \Rightarrow m &= \pm \frac{\mathbf{i}}{\sqrt{3}} \end{aligned}$$

General solution:

$$y = c_1 \cos \frac{x}{\sqrt{3}} + c_2 \sin \frac{x}{\sqrt{3}}.$$

16.

$$2y'' - 3y' + 4y = 0$$

Auxiliary equation:

$$\begin{aligned} 2m^3 - 3m + 4 &= 0 \\ \Rightarrow m &= \frac{3 \pm \sqrt{-23}}{4} = \frac{3}{4} \pm \frac{\sqrt{23}}{4}\,\mathbf{i} \end{aligned}$$

General solution:

$$y = e^{\frac{3}{4}x} \left(c_1 \cos \frac{\sqrt{23}}{4} x + c_2 \sin \frac{\sqrt{23}}{4} x \right)$$

17.

$$\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0$$

Auxiliary equation:

$$f(m) \equiv m^3 - m^2 - 4 = 0$$

Since $f(2) = 0, (m-2)$ is a factor of $f(m)$. Factorizing $f(m)$ gives

$$(m-2)(m^2+m+2) = 0.$$

Hence,

$$m = 2 \quad \text{or} \quad m = \frac{-1 \pm \sqrt{-7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}\,\mathbf{i}$$

General solution:

$$x = c_1 e^{2t} + e^{-\frac{1}{2}t}\left(c_2 \cos \frac{\sqrt{7}}{2}t + c_3 \sin \frac{\sqrt{7}}{2}t\right).$$

18.

$$y^{(4)} - 7y'' - 18y = 0$$

Auxiliary equation:

$$m^4 - 7m^2 - 18 = 0$$

Hence,

$$(m^2-9)(m^2+2) = 0$$

$$\Rightarrow m = \pm 3 \quad \text{or} \quad \pm\sqrt{2}\,\mathbf{i}$$

General solution:

$$y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos\sqrt{2}\,x + c_4 \sin\sqrt{2}\,x.$$

19.

$$y'' - 2y' + y = 0$$

Auxiliary equation:

$$\begin{aligned} m^2 - 2m + 1 &= 0 \\ (m-1)^2 &= 0 \\ m &= 1 \text{ (twice)} \end{aligned}$$

General solution:

$$y = c_1 e^x + c_2 x e^x$$

Hence,

$$y' = c_1 e^x + c_2(e^x + xe^x).$$

Substituting the initial values:

$$\begin{aligned} y(0) &= 5 \Rightarrow \quad c_1 = 5 \\ y'(0) &= 10 \Rightarrow \quad c_1 + c_2 = 10 \Rightarrow c_2 = 10 - c_1 = 5 \end{aligned}$$

The solution to the IVP is thus

$$y = 5e^x(1 + x).$$

20.

$$y'' - 2y' + 2y = 0$$

Auxiliary equation:

$$\begin{aligned} m^2 - 2m + 2 &= 0 \\ m = \frac{2 \pm \sqrt{-4}}{2} &= 1 \pm i \end{aligned}$$

General solution:

$$y = e^x(c_1 \cos x + c_2 \sin x)$$

Substituting the boundary values:

$$\begin{aligned} y(0) &= 1 \Rightarrow c_1 = 1 \\ y(\pi) &= 1 \Rightarrow -e^{\pi} c_1 = 1 \Rightarrow c_1 = -e^{-\pi} \neq 1 \end{aligned}$$

It is clear that this particular BVP does not have a solution.

21. Using the assumption $y = e^{mx}$ we obtain the auxiliary equation

$$y^{(4)} + y''' + y'' = 0$$

$$m^4 + m^3 + m^2 = m^2(m^2 + m + 1) = 0$$

with solutions $m_1 = 0$, $m_2 = 0$,

$$m_2 = \frac{-1 + \sqrt{1 - 4}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

$$m_3 = \frac{-1-\sqrt{1-4}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Thus we form the general solution

$$y(x) = c_1 + c_2 x + c_3 e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2}x + c_4 e^{-\frac{1}{2}x} \sin \frac{\sqrt{3}}{2}x.$$

22. From the roots $m_1 = 4$, $m_2 = m_3 = -5$ we reconstruct the auxiliary equation

$$(m-4)(m+5)(m+5) = (m-4)(m^2+10m+25) = m^3+6m^2-15m-100 = 0.$$

A corresponding equation would be

$$y''' + 6y'' - 15y' - 100y = 0.$$

Multiplying by an arbitrary constant $c \neq 0$ yields equivalent equations

$$c(y''' + 6y'' - 15y' - 100y) = 0.$$

Thus the answer is not unique, but it is unique up to the constant c.

23. Using the assumption $y = e^{mx}$ we obtain the auxiliary equation

$$y''' + 6y'' + y' - 34y = 0$$

$$\Rightarrow m^3 + 6m^2 + m - 34 = 0.$$

Since $y_1 = e^{-4x} \cos x$ is a solution we know that $m_1 = -4 + i$ is a solution for the auxiliary equation. Thus the complex conjugate $m_2 = -4 - i$ is also a solution. Thus we have the factor

$$(m - m_1)(m - m_2) = (m + 4 - i)(m + 4 + i) = m^2 + 8m + 17.$$

To find the last factor we perform long division:

	m	-2		
$m^2 + 8m + 17$	m^3	$+6m^2$	$+m$	-34
	$-(m^3$	$+8m^2$	$+17m)$	
		$-2m^2$	$-16m$	-34
		$-(-2m^2$	$-16m$	$-34)$
				0

thus $m_3 = 2$ and the general solution is

$$y(x) = c_1 e^{-4x} \cos x + c_2 e^{-4x} \sin x + c_3 e^{2x}.$$

24.

$$\begin{aligned} y''' - y &= 0 \\ &\Rightarrow m^3 - 1 = 0 \Rightarrow m^3 = 1 \\ &\Rightarrow m_1 = 1, m_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i} \text{ and } m_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i} \\ &\Rightarrow y_1 = e^x, y_2 = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), y_3 = e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right) \\ &\Rightarrow y(x) = c_1 e^x + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right). \end{aligned}$$

25.

$$y^{(5)} + 2y^{(4)} + 6y''' + 12y'' + 9y' + 18y = 0$$

$$\begin{aligned} &\Rightarrow (m+2)\left(m^2+3\right)\left(m^2+3\right) = 0 \\ &\Rightarrow m_1 = -2, m_2 = \sqrt{3}\mathbf{i}, m_3 = -\sqrt{3}\mathbf{i}, m_4 = \sqrt{3}\mathbf{i}, m_5 = -\sqrt{3}\mathbf{i} \\ &\left(\Rightarrow \alpha = 0, \beta = \sqrt{3} \Rightarrow e^{\alpha x} = 1\right) \end{aligned}$$

$$\Rightarrow \begin{cases} y_1 = e^{-2x} \\ y_2 = \cos\left(\sqrt{3}x\right) \\ y_3 = \sin\left(\sqrt{3}x\right) \\ y_4 = x\cos\left(\sqrt{3}x\right) \\ y_5 = x\sin\left(\sqrt{3}x\right) \end{cases}$$

$$\Rightarrow y(x) = c_1 e^{-2x} + c_2\cos\left(\sqrt{3}x\right) + c_3\sin\left(\sqrt{3}x\right) + c_4 x\cos\left(\sqrt{3}x\right) + c_5 x\sin\left(\sqrt{3}x\right).$$