TWK2A
Reduction of order
Solutions

1.

$$\begin{aligned} y_2 &= e^{5x}\int \frac{e^{\int 0dx}}{e^{10x}}dx = e^c e^{5x}\int e^{-10x}dx \\ &= \left(-\frac{e^c}{10}\right)e^{-5x} \quad ; \quad c \in \mathbb{R} \end{aligned}$$

and so $y_2 = e^{-5x}$ is a second solution, because a constant multiple of a solution is also a solution.

2.

$$\begin{aligned} y_2 &= x^4\int \frac{e^{\int \frac{7}{x}dx}}{x^8}dx = x^4\int \frac{e^{7\ln x}}{x^8}dx \\ &= x^4\int \frac{x^7}{x^8}dx = x^4\int \frac{1}{x}dx \\ &= x^4\ln x \end{aligned}$$

3.

$$y_2 = \ln x\int \frac{e^{-\int \frac{1}{x}dx}}{(\ln x)^2}dx = \ln x\int \frac{1}{x(\ln x)^2}dx$$

We use the substitution $u = \ln x$ ($\Rightarrow dx = xdu$) for the integration:

$$y_2 = u\int \frac{du}{u^2} = -1.$$

4.

$$\begin{aligned} y_2 &= x^2\cos(\ln x)\int \frac{e^{\int \frac{3}{x}dx}}{x^4\cos^2(\ln x)}dx = x^2\cos(\ln x)\int \frac{x^3}{x^4\cos^2(\ln x)}dx \\ &= x^2\cos(\ln x)\tan(\ln x) = x^2\sin(\ln x) \end{aligned}$$

We used the substitution $u = \ln x$ for the integration.

5. We have

$$y'' + \frac{1}{6}y' - \frac{1}{6}y = 0$$

and so

$$P(x) = \frac{1}{6}.$$

Hence,

$$\begin{aligned} y_2 &= e^{\frac{x}{3}} \int \frac{e^{-\int \frac{1}{6}dx}}{\left(e^{\frac{x}{3}}\right)^2} dx = e^{\frac{x}{3}} \int \frac{e^{-\frac{x}{6}}}{e^{\frac{2x}{3}}} dx \\ &= e^{\frac{x}{3}} \int e^{-\frac{5x}{6}}\, dx = -\frac{6}{5} e^{\frac{x}{3}} e^{-\frac{5x}{6}} \\ &= -\frac{6}{5} e^{-\frac{x}{2}} \end{aligned}$$

6. We have

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\, dx$$

and so

$$\begin{aligned} y_2' &= y_1' \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\, dx + y_1 \frac{d}{dx} \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\, dx \\ &= y_1' \frac{y_2}{y_1} + y_1 \frac{e^{-\int P(x)dx}}{y_1^2} \\ &= \frac{1}{y_1}\left(y_1' y_2 + e^{-\int P(x)dx}\right) \end{aligned}$$

Hence,

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} y_1 & y_2 \\ y_1' & \frac{1}{y_1}\left(y_1' y_2 + e^{-\int P(x)dx}\right) \end{vmatrix} \\ &= y_1' y_2 + e^{-\int P(x)dx} - y_1' y_2 \\ &= e^{-\int P(x)dx} \\ &\neq 0 \qquad \forall x \end{aligned}$$

7. We have

$$ay'' + by' + cy = 0 \Rightarrow y'' + \left(\frac{b}{a}\right) y' + \left(\frac{c}{a}\right) y = 0 \quad \left(\Rightarrow P = \frac{b}{a}\right)$$

and so

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int P dx}}{(y_1)^2} dx \\
&= e^{-\frac{b}{2a}x} \int \frac{e^{-\int \frac{b}{a} dx}}{e^{-\frac{b}{a}x}} dx \\
&= e^{-\frac{b}{2a}x} \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}} dx \\
&= e^{-\frac{b}{2a}x} \int dx \\
&= xe^{-\frac{b}{2a}x} \\
&= xy_{1.}
\end{aligned}$$