TWK2A
Section 4.1
Solutions

1. We have

$$\begin{aligned} 4y &= 4\left(c_1e^{4x} + c_2e^{-x}\right) = 4c_1e^{4x} + 4c_2e^{-x} \\ 3y' &= 3\left(4c_1e^{4x} - c_2e^{-x}\right) = 12c_1e^{4x} - 3c_2e^{-x} \\ y'' &= 16c_1e^{4x} + c_2e^{-x} \end{aligned}$$

   so that

$$y'' - 3y' - 4y = 0$$

   and y is indeed a solution. The initial values give

$$\begin{aligned} y(0) = 1 &\;\Rightarrow\; c_1 + c2 = 1 \\ y'(0) = 2 &\;\Rightarrow\; 4c_1 - c_2 = 2 \end{aligned}$$

   Solving the equations yields $c_1 = \frac{3}{5}$ and $c_2 = \frac{2}{5}$. Thus the solution to the initial value problem is $y = \frac{3}{5}e^{4x} - \frac{2}{5}e^{-x}$.

2. We have

$$\begin{aligned} y' &= -c_2 \sin x + c_3 \cos x \\ y''' &= c_2 \sin x - c_3 \cos x \end{aligned}$$

   so that

$$y''' + y' = 0$$

   and y is a solution. The initial values give

$$\begin{aligned} y(\pi) = 0 &\;\Rightarrow\; c_1 - c_2 = 0 \\ y'(\pi) = 2 &\;\Rightarrow\; -c_3 = 2 \\ y''(\pi) = -1 &\;\Rightarrow\; c_2 = -1 \end{aligned}$$

   Solving the equations yields $c_1 = -1$, $c_2 = -1$ and $c_3 = -2$. Thus the solution to the initial value problem is

$$y = -1 - \cos x - 2 \sin x.$$

3.

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\ y'(0) = 1 \quad &\Rightarrow \quad c_1 = 1 \end{aligned}$$

Thus, we have a contradiction. From the equation

$$xy'' - y' = 0$$

we use $a_2(x) = x$ which is zero at $x = 0$, i.e. from theorem 4.1 we cannot guarantee the existence of a solution on any interval including 0 and thus not for $(-\infty, \infty)$.

4.

$$y' = c_1 e^x \cos x - c_1 e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x.$$

a)

$$\begin{aligned} y(0) = 1 \quad &\Rightarrow \quad c_1 = 1 \\ y'(\pi) = 0 \quad &\Rightarrow \quad -c_1 - c_2 = 0 \\ c_1 = 1 \quad & \quad c_2 = -1 \end{aligned}$$

b)

$$\begin{aligned} y(0) = 1 \quad &\Rightarrow \quad c_1 = 1 \\ y(\pi) = -1 \quad &\Rightarrow \quad -c_1 = -e^{-\pi} \\ &\Rightarrow \quad \text{contradiction} \end{aligned}$$

c)

$$\begin{aligned} y(0) = 1 \quad &\Rightarrow \quad c_1 = 1 \\ y(\frac{\pi}{2}) = 1 \quad &\Rightarrow \quad c_2 = e^{-\frac{\pi}{2}} \\ c_1 = 1 \quad & \quad c_2 = e^{-\frac{\pi}{2}} \end{aligned}$$

d)

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\ y(\pi) = 0 \quad &\Rightarrow \quad -c_1 = 0 \\ c_1 = 0 \quad & \quad c_2 \in \mathbf{R} \end{aligned}$$

5.
$$f_3(x) = 4f_1(x) - 3f_2(x).$$
Thus f_1, f_2 and f_3 are linearly dependent.

6.
$$f_1(x) = 5f_2(x) + 5f_3(x).$$
Thus f_1, f_2 and f_3 are linearly dependent.

7. It is simple matter to verify that the functions are solutions of the DE. We now calculate the Wronskian
$$\begin{aligned} W(x^3, x^4) &= \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} \\ &= 4x^6 - 3x^6 = x^6 \\ x^6 = 0 &\Leftrightarrow x = 0 \end{aligned}$$
The two functions x^3 and x^4 are thus linearly independent on $(0, \infty)$, and so they form a fundamental set of solutions for the equation on $(0, \infty)$. The general solution is
$$y = c_1x^3 + c_2x^4.$$

8. We have
$$\begin{aligned} y''_{p_1} - 6y_{p_1} + 5y_{p_1} &= 12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x} \\ y''_{p_2} - 6y_{p_2} + 5y_{p_2} &= 2 - 12x - 18 + 5x^2 + 15x = 5x^2 + 3x - 16. \end{aligned}$$
Since the equation is linear we find that a solution to the first equation is $y_{p_1} + y_{p_2}$.
$$-10x^2 - 6x + 32 + e^{2x} = -2(5x^2 + 3x + 16) - \frac{1}{9}(-9e^{2x})$$
so that a solution to the second equation is
$$-2y_{p_2} - \frac{1}{9}y_{p_1}.$$

9. **(a)** Obviously a constant solution will be sufficient. Let $y = k$, then the equation becomes $2k = 10$ and $y = 5$ is a solution. **(b)** We make the assumption $y = kx$ (since the equation is second order $y'' = 0$), so the equation becomes $2kx = -4x$ and we find the solution $y = -2x$. **(c)** $y = -2x + 5$. **(d)** $y = 4x + \frac{5}{2}$.

10.

$$\begin{aligned}\cosh x &= \frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\\ \sinh x &= \frac{1}{2}e^{x}-\frac{1}{2}e^{-x}\end{aligned}$$

Thus $\cosh x$ and $\sinh x$ are expressed as linear combinations of the solutions. We know that linear homogenous equations have the property that the sum of solutions are also solutions to the equation.

11. (a)

$$x=-1;\quad y=0\Rightarrow c_1+c_2+3=0 \tag{1}$$
$$x=1;\quad y=4\Rightarrow c_1+c_2+3=4 \tag{2}$$

and $(2)-(1)$ implies $0=4$.

No solution can satisfy these boundary conditions.

(b)

$$x=0;\quad y=1\Rightarrow 3=1$$

No solution can satisfy this boundary condition.

(c)

$$x=0;\quad y=3\Rightarrow 3=3$$
$$x=1;\quad y=0\Rightarrow c_1+c_2+3=0$$

These boundary conditions allow a family of solutions, viz. those for which $c_1+c_2=3$.

(d)

$$x=1;\quad y=3\Rightarrow c_1+c_2+3=3\Rightarrow c_1+c_2=0 \tag{3}$$
$$x=2;\quad y=15\Rightarrow 4c_1+16c_2+3=15\Rightarrow c_1+4c_2=3 \tag{4}$$

$(4)-(3)\Rightarrow 3c_2=3\Rightarrow c_2=1$.

Substitute in (3): $c_1 + 1 = 0 \Rightarrow c_1 = -1$.

These boundary values admit one solution:

$$y = -x^2 + x^4 + 3$$

12.

$$W(f_1, f_2, f_3) = \begin{vmatrix} 0 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = 0$$

for all x. Thus the functions are linearly dependent..

13. We calculate the Wronskian on the sub-interval $(0, \infty)$:

$$W(f_1, f_2) = \begin{vmatrix} 2+x & 2+x \\ 1 & 1 \end{vmatrix} = 0$$

On $(-\infty, 0)$ we have

$$W(f_1, f_2) = \begin{vmatrix} 2+x & 2-x \\ 1 & -1 \end{vmatrix} = 2x \neq 0$$

Therefore (f_1, f_2) is linearly independent on $(-\infty, 0)$ and linearly dependent on $(0, \infty)$. (Since the derivatives of $|x|$ do not exist for $x = 0$, it implies that no solution is possible for $x = 0$.)

14. For $y = e^{\frac{x}{2}}$ we have

$$y' = \frac{1}{2}e^{\frac{x}{2}} \quad ; \quad y'' = \frac{1}{4}e^{\frac{x}{2}}$$

and so

$$\begin{aligned} 4y'' - 4y' + y &= 4\left(\frac{1}{4}\right)e^{\frac{x}{2}} - 4\left(\frac{1}{2}\right)e^{\frac{x}{2}} + e^{\frac{x}{2}} \\ &= (1 - 2 + 1)e^{\frac{x}{2}} \\ &= 0 \end{aligned}$$

For

$$y = xe^{\frac{x}{2}}$$

we have

$$y' = e^{\frac{x}{2}} + \frac{1}{2}xe^{\frac{x}{2}}$$

and

$$y'' = \frac{1}{2}e^{\frac{x}{2}} + \frac{1}{2}e^{\frac{x}{2}} + \frac{1}{4}xe^{\frac{x}{2}} = \left(1 + \frac{1}{4}x\right)e^{\frac{x}{2}}$$

and so

$$\begin{aligned} 4y'' - 4y' + y &= 4\left(1 + \frac{1}{4}x\right)e^{\frac{x}{2}} - 4\left(1 + \frac{1}{2}x\right)e^{\frac{x}{2}} + xe^{\frac{x}{2}} \\ &= (4 + x - 4 - 2x + x)\,e^{\frac{x}{2}} \\ &= 0 \end{aligned}$$

Thus both functions are solutions of the DE. Furthermore, we have

$$\begin{aligned} W\left(e^{\frac{x}{2}}, xe^{\frac{x}{2}}\right) &= \begin{vmatrix} e^{\frac{x}{2}} & xe^{\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & \left(1 + \frac{1}{2}x\right)e^{\frac{x}{2}} \end{vmatrix} \\ &= \left(1 + \frac{1}{2}x\right)e^{x} - \frac{1}{2}xe^{x} \\ &= e^{x} \end{aligned}$$

Since $e^x \neq 0$ on $(-\infty, \infty)$, the solutions are linearly independent and so the general solution is given by $y = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$.

15. For

$$y = \cos(\ln x)$$

we have

$$y' = -\frac{1}{x}\sin(\ln x)$$

and

$$y'' = \frac{1}{x^2}\sin(\ln x) - \frac{1}{x^2}\cos(\ln x)$$

and thus

$$\begin{aligned} x^2 y'' + xy' + y &= \sin(\ln x) - \cos(\ln x) - \sin(\ln x) + \cos(\ln x) \\ &= 0. \end{aligned}$$

For

$$y = \sin(\ln x)$$

we have

$$y' = \frac{1}{x}\cos(\ln x)$$

and

$$y'' = -\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x)$$

and thus

$$\begin{aligned} x^2y'' + xy' + y &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0. \end{aligned}$$

Furthermore we have

$$\begin{aligned} W\left(\cos(\ln x), \sin(\ln x)\right) &= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} \\ &= \frac{1}{x}\cos^2(\ln x) + \frac{1}{x}\sin^2(\ln x) \\ &= \frac{1}{x} \end{aligned}$$

Since $\frac{1}{x} \neq 0$ on $(0, \infty)$ the two solutions are linearly independent and the general solution is given by

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

16. We first consider

$$y_c = c_1 \cos x + c_2 \sin x. \tag{5}$$

Then

$$y_c' = -c_1 \sin x + c_2 \cos x$$

and

$$y_c'' = -c_1 \cos x - c_2 \sin x.$$

Thus

$$y_c'' + y_c = 0.$$

Furthermore,

$$\begin{aligned} W(\cos x, \sin x) &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x = 1 \\ &\neq 0 \end{aligned}$$

Therefore (5) constitutes the complementary function. We next consider

$$y_p = x\sin x + \cos x \ln(\cos x) \tag{6}$$

Then

$$\begin{aligned} y_p' &= \sin x + x\cos x - \sin x\ln(\cos x) + \cos x\left(\frac{-\sin x}{\cos x}\right) \\ &= x\cos x - \sin x\ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \cos x - x\sin x - \cos\ln(\cos x) - \sin x\left(\frac{-\sin x}{\cos x}\right) \\ &= \cos x - x\sin x - \cos x\ln(\cos x) + \frac{\sin^2 x}{\cos x} \end{aligned}$$

Therefore

$$\begin{aligned} y_p'' + y_p &= \cos x - x\sin x - \cos x\ln(\cos x) + \frac{\sin^2 x}{\cos x} \\ &\qquad + x\sin x + \cos x\ln(\cos x) \\ &= \cos x + \frac{\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos x} \\ &= \sec x, \end{aligned}$$

and (6) constitutes the particular solution.

17. We have

$$\begin{aligned} y &= c_1 x^{\frac{1}{2}} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x \\ y' &= -\frac{1}{2}c_1 x^{-\frac{3}{2}} - c_2 x^{-2} + \frac{2}{15}x - \frac{1}{6} \\ y'' &= \frac{3}{4}c_1 x^{-\frac{5}{2}} + 2c_2 x^{-3} + \frac{2}{15}. \end{aligned}$$

We substitute these expressions in the ODE.

$$\begin{aligned}
& 2x^2y'' + 5xy' + y \\
= \;& \frac{3}{2}c_1x^{-\frac{1}{2}} + 4c_2x^{-1} + \frac{4}{15}x^2 - \frac{5}{2}c_1x^{-\frac{1}{2}} - 5c_2x^{-1} + \frac{2}{3}c^2 - \frac{5}{6}x \\
& \qquad + c_1x^{-\frac{1}{2}} + c_2x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x \\
= \;& x^2 - x
\end{aligned}$$

Therefore, y is a solution of the ODE. To establish that it is a general solution, we consider the two members of the complementary function

$$\begin{aligned}
W(x^{-\frac{1}{2}}, x^{-1}) &= \begin{vmatrix} x^{-\frac{1}{2}} & x^{-1} \\ -\frac{1}{2}x^{-\frac{3}{2}} & -x^{-2} \end{vmatrix} \\
&= -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{5}{2}} \\
&= -\frac{1}{2}x^{-\frac{5}{2}} \\
&\neq 0 \;\; \forall x.
\end{aligned}$$

Since $x^{-\frac{1}{2}}$ is only defined on $(0, \infty)$, y is a general solution on $(0, \infty)$.

18. Since

$$f_1 = \cos 2x = 2\cos^2 x - 1 = 2f_3 - f_2$$

the functions are linearly dependent. In other words we have the solution $c_1 = c_2 = -1$ and $c_3 = 2$.

OR

$$\begin{aligned}
W(\cos 2x, 1, \cos^2 x) &= \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & -\sin 2x \\ -4\cos 2x & 0 & -2\cos 2x \end{vmatrix} \\
&= 4\sin 2x\cos 2x - 4\sin 2x\cos 2x = 0.
\end{aligned}$$

Since the Wronskian is zero the functions are linearly dependent.

NOTE: We have used $\frac{d}{dx}\cos^2 x = -2\cos x\sin x = -\sin 2x$.

19. Consider

$$W(y_1, y_2, \ldots, y_{n+1}) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n+1} \\ y_1' & y_2' & \cdots & y_{n+1}' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n+1}^{(n)} \end{vmatrix}.$$

Each $y_j,\ j = 1, 2, \ldots, n+1$ satisifies the nth order homogeneous linear differential equation with constant coefficients $a_0, a_1, \ldots, a_n$ with $a_n \neq 0$

$$a_n y_j^{(n)} + a_{n-1} y_j^{(n-1)} + \ldots + a_0 y_j = 0$$

i.e.

$$y_j^{(n)} = -\frac{1}{a_n}\left(a_{n-1} y_j^{(n-1)} + \ldots + a_0 y_j\right).$$

Thus each $y_j^{(n)}$ is a linear combination of $y_j, y_j', \ldots, y_j^{(n-1)}$ and consequently the last row of the determinant $W(y_1, y_2, \ldots, y_{n+1})$ is a linear combination of the first n rows in the determinant $W(y_1, y_2, \ldots, y_{n+1})$. It follows that $W(y_1, y_2, \ldots, y_{n+1}) = 0$. Thus $y_1, y_2, \ldots, y_{n+1}$ are linearly dependent. Consequently n linearly independent solutions are required to form the general solution, since any more solutions cannot be linearly independent of the first n solutions and thus will not form part of the fundamental set of solutions.

20.

$$\begin{aligned} W\left(e^x, e^{-2x}\right) &= \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} = \left(e^x\right)\left(-2e^{-2x}\right) - \left(e^{-2x}\right)\left(e^x\right) \\ &= -2e^{-x} - e^{-x} \\ &= -3e^{-x} \quad \text{which is nonzero everywhere on } (0, \infty) \end{aligned}$$

21.

$$\begin{aligned} W\left(1, \cos x, \sin x\right) &= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\ &= \sin^2 x + \cos^2 x \\ &= 1 \quad \text{which is obviously nonzero everywhere on } (-\infty, \infty) \end{aligned}$$