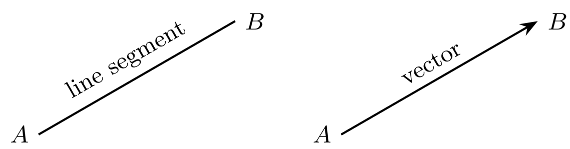
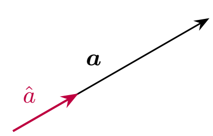
Vectors

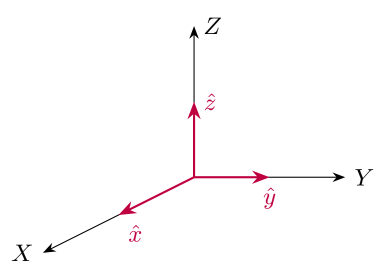
- A *scalar* is a number which is used to represent some quantity, for example magnitudes (which is the *size* of a mathematical object).
- Scalars are indicated by regular Roman text, for example a or x, or by using Greek alphabet letters, for example α or λ.
- A *vector* is a mathematical object with magnitude and direction.
- Geometrically, vectors are indicated by an arrow; vectors may also be represented by a directed line segment, that is, a line segment with a specified *initial point* A and a specified *terminal point* B and indicated as $\overline{AB}$.



- Algebraically, and in text, vectors are indicated by bold face letters, for example $\boldsymbol{a}$; by a bar over the letter, for example $\overline{a}$; by an arrow over the letter, for example $\vec{a}$.
- The magnitude of a vector $\boldsymbol{a}$ is a scalar value and is indicated by writing the vector in regular Roman text, for example a, or by enclosing the vector in two vertical bars, for example $|\boldsymbol{a}|$.
- An *unit vector* is a vector which has magnitude equal to one. If $\boldsymbol{a}$ is a vector with magnitude $|\boldsymbol{a}|$, then the unit vector with the same direction as $\boldsymbol{a}$ is indicated by $\hat{a} = \boldsymbol{a}/a$.

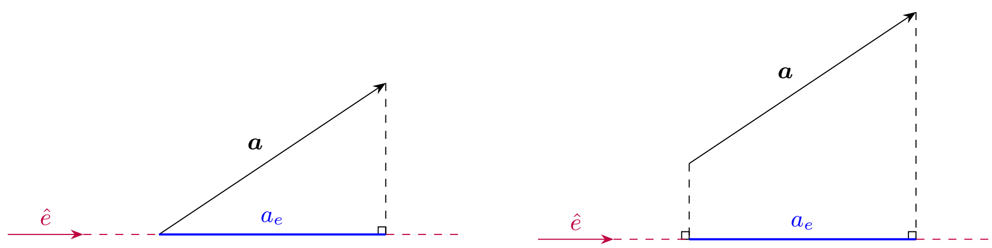


- The unit vectors in the same direction has the positive axes of the Cartesian reference system are indicated by $\hat{x}$, $\hat{y}$, and $\hat{z}$.

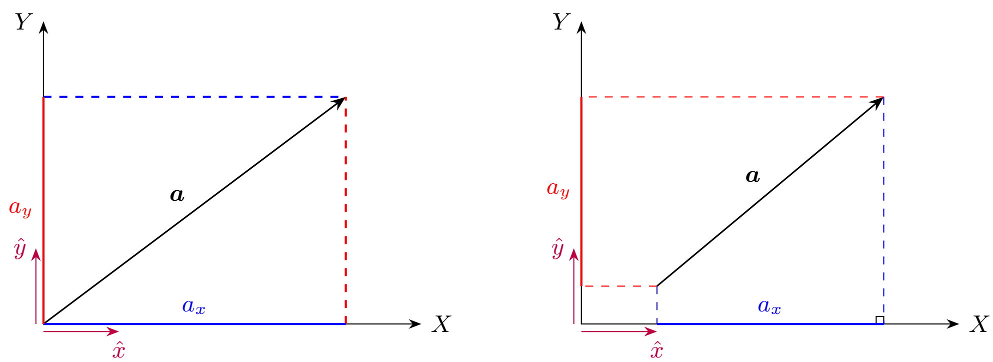


Vector components

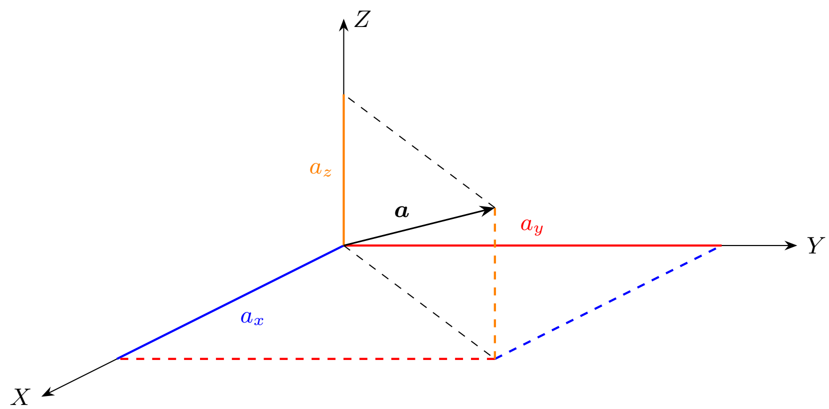
- The *component* of a vector in a given direction can be thought of as the magnitude of the vector measured in that direction.
- In the following figure, a_e is the component of the vector $\boldsymbol{a}$ in the e direction.

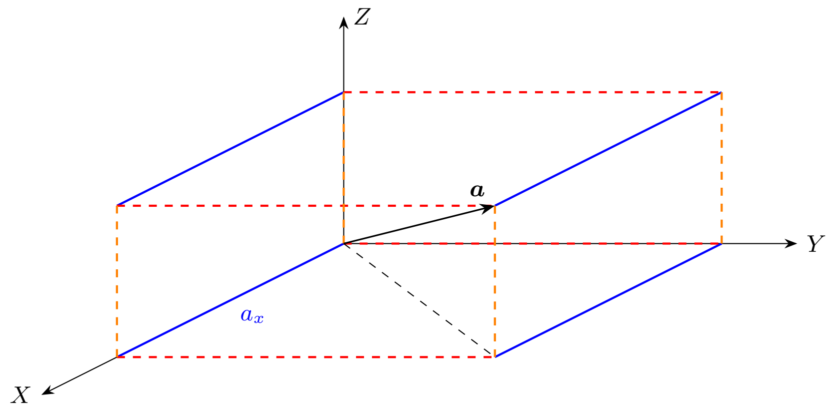


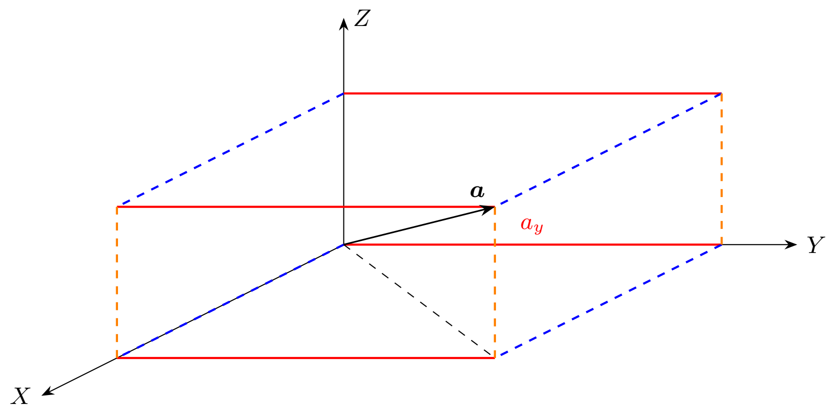
- The components a_x and a_y of a vector $\boldsymbol{a}$ in the two-dimensional Cartesian reference system are shown in the following figure:

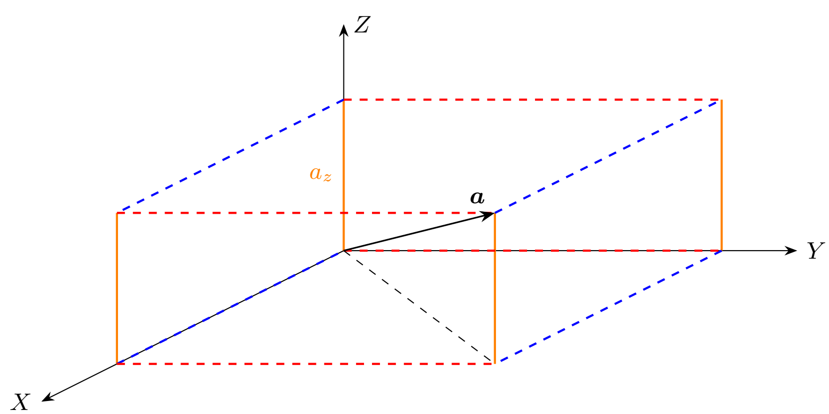


- The components a_x, a_y, and a_z of a vector $\boldsymbol{a}$ in the three-dimensional Cartesian reference system are shown in the following figures:



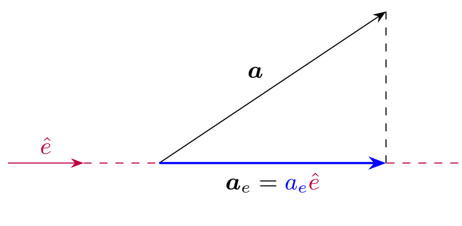

Z
a
Y
a_x
X

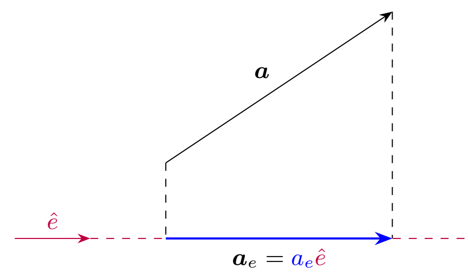

Z
a
a_y
Y
X


Z
a_z
a
Y
X

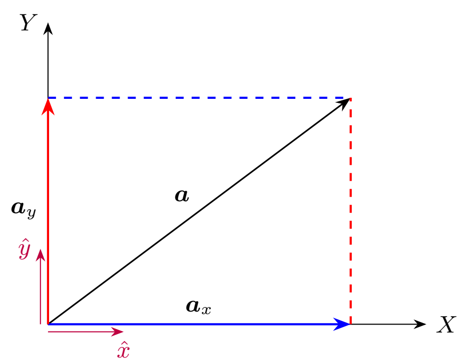
Vector projections

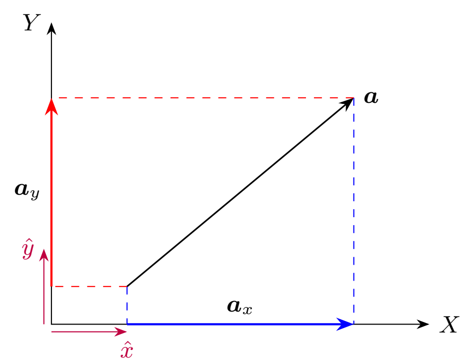
- The *projection* of a vector $\boldsymbol{a}$ in the e direction is indicated by $\boldsymbol{a}_e$ and defined as $\boldsymbol{a}_e = a_e\hat{e}$, that is,
  - its magnitude is a_e, the component of $\boldsymbol{a}$ in the e direction; and
  - its direction is the same as that of the vector $\hat{e}$.





- Note the projection $\boldsymbol{a}_x$ of the vector $\boldsymbol{a}$ in the x direction and the projection $\boldsymbol{a}_y$ of the vector $\boldsymbol{a}$ in the y direction (in two-dimensional Cartesian reference system) in the following figures:

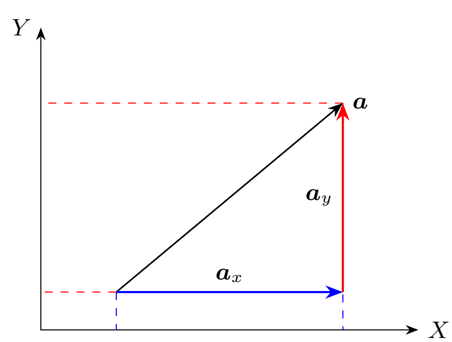




Clearly, we have

$$\boldsymbol{a}_x = a_x\hat{x} \qquad \text{and} \qquad \boldsymbol{a}_y = a_y\hat{y}.$$

We could also translate these projection vectors as follows:

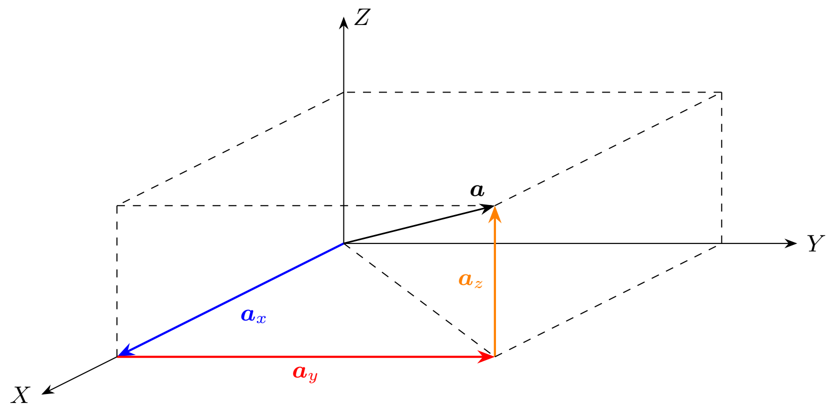


By the laws of vector addition it now follows that

$$\boldsymbol{a} = \boldsymbol{a}_x + \boldsymbol{a}_y = a_x\hat{x} + a_y\hat{y}.$$

This is called the *component form* of the vector $\boldsymbol{a}$.

- Consider now the following figure of a vector in the three-dimensional Cartesian reference system:

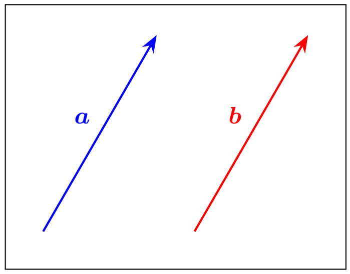


Again, it follows from vector addition that

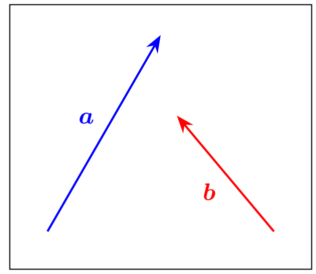
$$\boldsymbol{a} = \boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}.$$

Vector algebra

- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *equal*, indicated by $\boldsymbol{a} = \boldsymbol{b}$, if they have the same direction and magnitude.



$\boldsymbol{a} = \boldsymbol{b}$

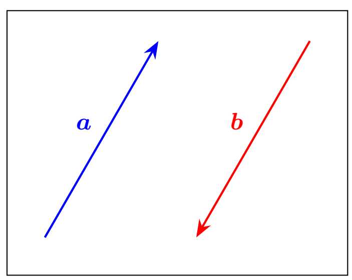


$\boldsymbol{a} \neq \boldsymbol{b}$

If $\boldsymbol{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ and $\boldsymbol{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$, then $\boldsymbol{a} = \boldsymbol{b}$ if and only if

$$a_x = b_x, \quad a_y = b_y, \quad a_z = b_z.$$

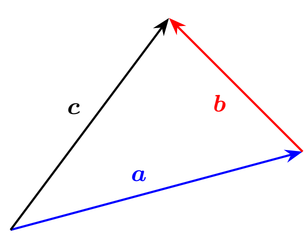
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that have the same magnitude but opposite (and parallel) are called *opposites* of each other, indicated by $\boldsymbol{a} = -\boldsymbol{b}$.



$\boldsymbol{a} = -\boldsymbol{b}$

In the figure above, $\boldsymbol{a}$ is also called the *negative* of $\boldsymbol{b}$.

- The sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, called the *vector sum*, is a third vector $\boldsymbol{c}$. This sum is indicated by $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$.



$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$

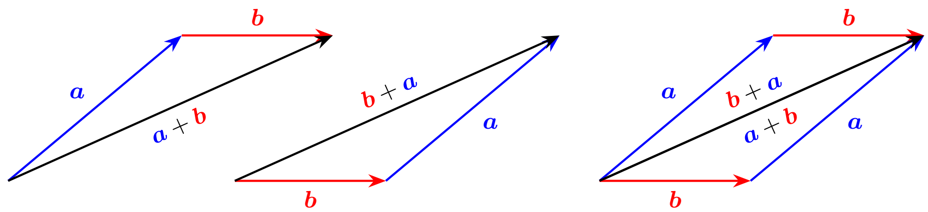
If $\boldsymbol{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, $\boldsymbol{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$ and $\boldsymbol{c} = c_x\hat{x} + c_y\hat{y} + c_z\hat{z}$, then $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ if and only if

$$c_x = a_x + b_x, \quad c_y = a_y + b_y, \quad c_z = a_z + b_z.$$

The algebraic properties of the vector sum are:

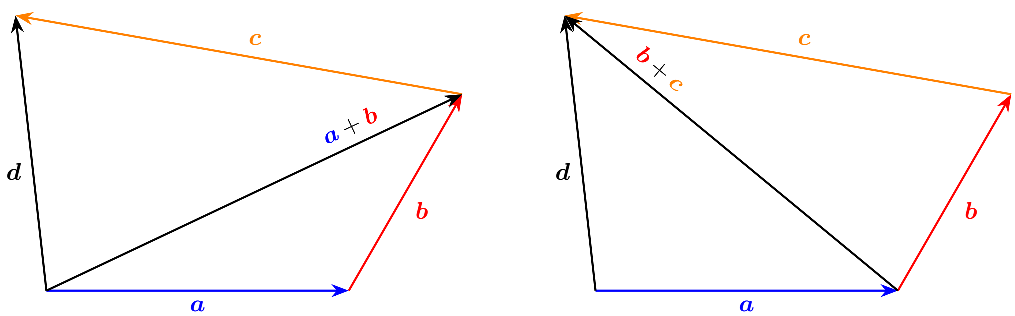
- the vector sum is *closed*, that is, the sum is a vector itself;
- the vector sum is *commutative*, that is,

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a};$$



- the vector sum is *associative*, that is,

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}).$$



$\boldsymbol{d} = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$ $\qquad$ $\boldsymbol{d} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$

- The *null vector*, indicated by $\mathbf{0}$, is obtained from the vector sum of a vector and its own negative, that is,

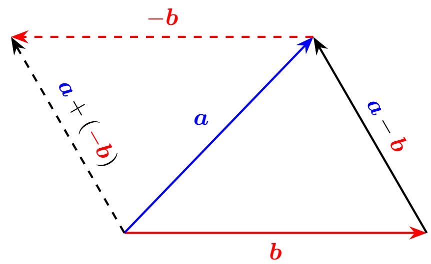
$$\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}.$$

The null vector is a vector with magnitude which is zero and the direction of which cannot be determined. Any vector remains unchanged when the null vector is added to it, that is,

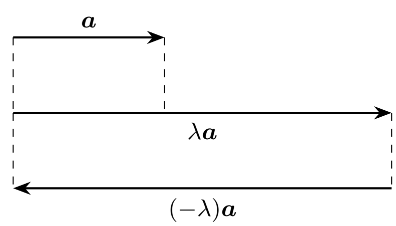
$$\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}.$$

- The difference between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, called the *vector difference* and indicated by $\boldsymbol{a} - \boldsymbol{b}$, is defined as

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}).$$



- The product of a scalar $\lambda > 0$ and a vector $\boldsymbol{a}$, indicated by $\lambda\boldsymbol{a}$, is a vector defined as follows:
  - $\lambda\boldsymbol{a}$ is parallel to $\boldsymbol{a}$, indicated as $\lambda\boldsymbol{a} \parallel \boldsymbol{a}$, that is, it has the same direction as $\boldsymbol{a}$;
  - the magnitude of $\lambda\boldsymbol{a}$ is given by $|\lambda\boldsymbol{a}| = \lambda|\boldsymbol{a}|$.



If $\boldsymbol{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, then

$$\begin{aligned}\lambda\boldsymbol{a} &= \lambda(a_x\hat{x} + a_y\hat{y} + a_z\hat{z}) \\ &= (\lambda a_x)\hat{x} + (\lambda a_y)\hat{y} + (\lambda a_z)\hat{z}.\end{aligned}$$

Note that

$$(-\lambda)\boldsymbol{a} = -(\lambda\boldsymbol{a})$$

and

$$0\boldsymbol{a} = \boldsymbol{0}.$$

The algebraic properties of this product are:

$$\lambda(\mu\boldsymbol{a}) = (\lambda\mu)\boldsymbol{a} \qquad \text{(associativity)}$$
$$\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b} \qquad \text{(distributivity)}$$