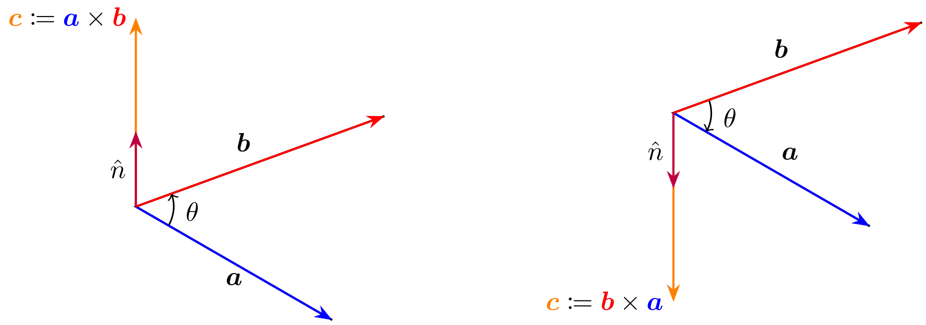
Vector product

- The *vector product* of the vector $\boldsymbol{a}$ and the vector $\boldsymbol{b}$ is a *vector*.
- It is denoted by $\boldsymbol{a} \times \boldsymbol{b}$.
- Its magnitude equal to the product of the *magnitudes of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ and the *sine of the smallest angle* θ *between their forward directions*, that is,

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta; \tag{1}$$

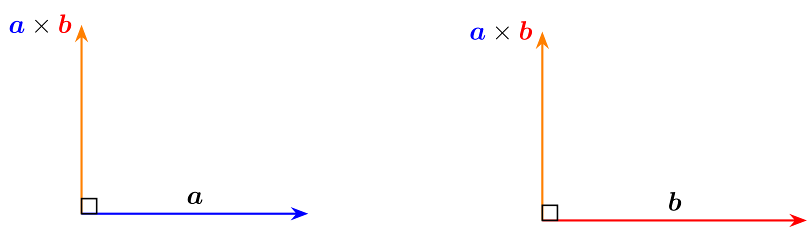
- Its direction is determined as follows: if the angle θ is measured from the vector $\boldsymbol{a}$ to the vector $\boldsymbol{b}$ and we curl our fingers in the direction of this measurement, then the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is in the same direction as which our thumb is pointing. (In the following figures, the direction of the vector obtained by the vector product is in the same direction as $\hat{n}$.)

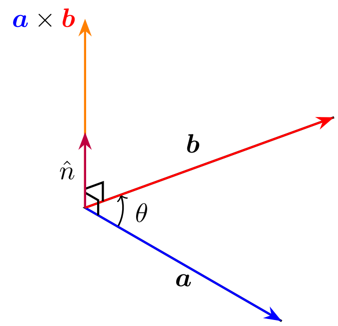


- Thus, the vector product of the vector $\boldsymbol{a}$ and the vector $\boldsymbol{b}$ is given by

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a} \times \boldsymbol{b}|\hat{n} = (|\boldsymbol{a}||\boldsymbol{b}| \sin\theta)\,\hat{n}. \tag{2}$$

- The vector obtained by the vector product is *perpendicular to both vector* $\boldsymbol{a}$ *and vector* $\boldsymbol{b}$*!*





- If $\boldsymbol{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ and $\boldsymbol{b} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}$, then

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{x} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \hat{y} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{z} \\ &= (a_y b_z - a_z b_y)\, \hat{x} + (a_z b_x - a_x b_z)\, \hat{y} + (a_x b_y - a_y b_x)\, \hat{z}, \end{aligned} \tag{3}$$

  where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Algebraic properties

- The vector product is *closed* (this means that the product yields a vector).
- The vector product is *anticommutative*, that is,

$$\boldsymbol{b} \times \boldsymbol{a} = -\left(\boldsymbol{a} \times \boldsymbol{b}\right).$$

- The vector product is *associative with respect to multiplication of a scalar and a vector*, that is,

$$(\lambda \boldsymbol{a}) \times (\mu \boldsymbol{b}) = (\lambda \mu)(\boldsymbol{a} \times \boldsymbol{b}).$$

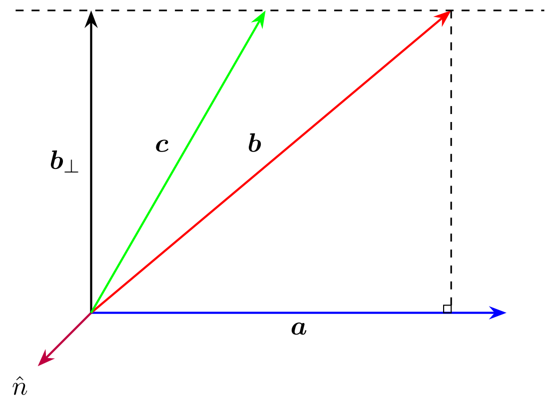
  Note that the vector product is not associative with respect to another vector product, that is,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}.$$

- The vector product is *distributive*, that is,

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}.$$

- The vector product of $\boldsymbol{a}$ and vectors with identical projections perpendicular to $\boldsymbol{a}$, are equal. Consider the following figure.



Then,

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}_\perp$$

and

$$\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c}_\perp.$$

However, from the figure it follows that $\boldsymbol{b}_\perp = \boldsymbol{c}_\perp$ and we conclude that

$$\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}.$$

(*For this special instance only!*)

Special cases

- If $\boldsymbol{a}$ is *parallel* to $\boldsymbol{b}$ (meaning the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to $0°$), then

  $$|\boldsymbol{a} \times \boldsymbol{b}| = 0$$

  and thus

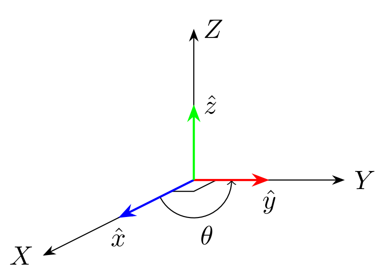
  $$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}.$$

  A similar result is true when the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to $180°$ (in this instance, we say that $\boldsymbol{a}$ is *antiparallel* to $\boldsymbol{b}$).

- The *vector product of* $\boldsymbol{a}$ *with itself* will yield the *null vector*, that is,

  $$\boldsymbol{a} \times \boldsymbol{a} = (|\boldsymbol{a}||\boldsymbol{a}| \sin 0°)\hat{n} = \boldsymbol{0}.$$

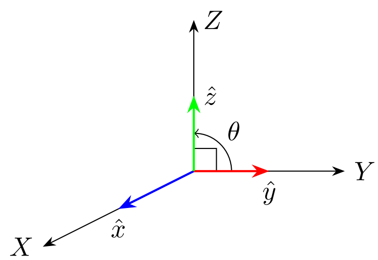
- Let us now calculate the vector product of the Cartesian unit vectors. Consider the following figure.



Note that $\theta = 90°$. It follows that

$$\hat{x} \times \hat{y} = \hat{z}.$$

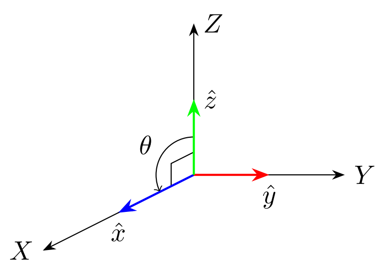
Consider now the following figure.



Again $\theta = 90°$ and we obtain

$$\hat{y} \times \hat{z} = \hat{x}.$$

Lastly, consider the following figure.



Again $\theta = 90°$ and we obtain

$$\hat{z} \times \hat{x} = \hat{y}.$$

Considering the anticommutativity property of the vector product, we note that

$$\hat{y} \times \hat{x} = -\hat{z}, \qquad \hat{z} \times \hat{y} = -\hat{x}, \qquad \hat{x} \times \hat{z} = -\hat{y}.$$

Finally, we should note that

$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = \mathbf{0}.$$