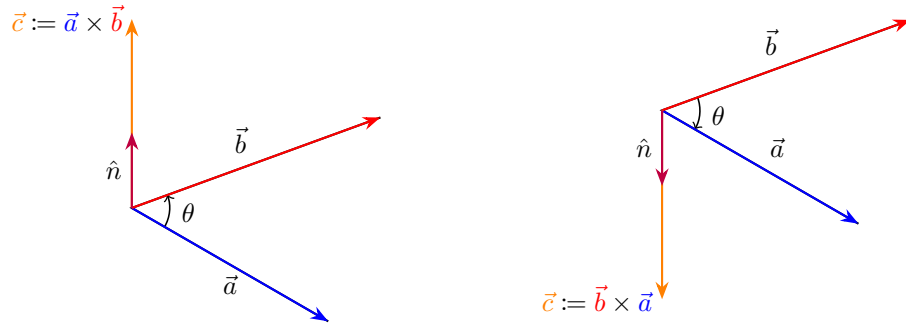


Vector product

- The *vector product* of the vector \vec{a} and the vector \vec{b} is a *vector*.
- It is denoted by $\vec{a} \times \vec{b}$.
- Its magnitude equal to the product of the *magnitudes of \vec{a} and \vec{b}* and the *sine of the smallest angle θ between their forward directions*, that is,

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta; \quad (1)$$

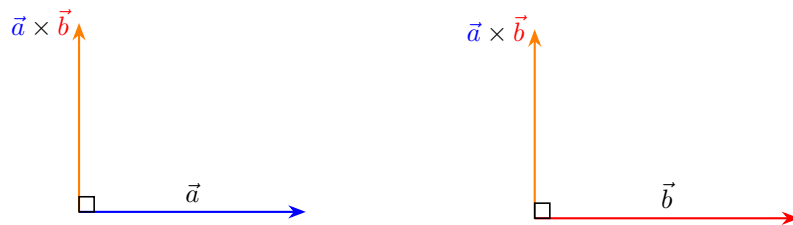
- Its direction is determined as follows: if the angle θ is measured from the vector \vec{a} to the vector \vec{b} and we curl our fingers in the direction of this measurement, then the vector product $\vec{a} \times \vec{b}$ is in the same direction as which our thumb is pointing. (In the following figures, the direction of the vector obtained by the vector product is in the same direction as \hat{n} .)

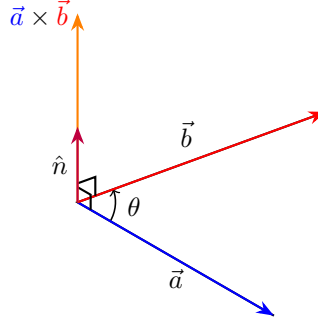


- Thus, the vector product of the vector \vec{a} and the vector \vec{b} is given by

$$\vec{c} = \vec{a} \times \vec{b} = |\vec{a} \times \vec{b}| \hat{n} = (|\vec{a}||\vec{b}| \sin \theta) \hat{n}. \quad (2)$$

- The vector obtained by the vector product is *perpendicular to both vector \vec{a} and vector \vec{b}* !





- If $\vec{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ and $\vec{b} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}$, then

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{x} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \hat{y} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{z} \\ &= (a_y b_z - a_z b_y) \hat{x} + (a_z b_x - a_x b_z) \hat{y} + (a_x b_y - a_y b_x) \hat{z}, \end{aligned} \quad (3)$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Algebraic properties

- The vector product is *closed* (this means that the product yields a vector).
- The vector product is *anticommutative*, that is,

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

- The vector product is *associative with respect to multiplication of a scalar and a vector*, that is,

$$(\lambda \vec{a}) \times (\mu \vec{b}) = (\lambda \mu) (\vec{a} \times \vec{b}).$$

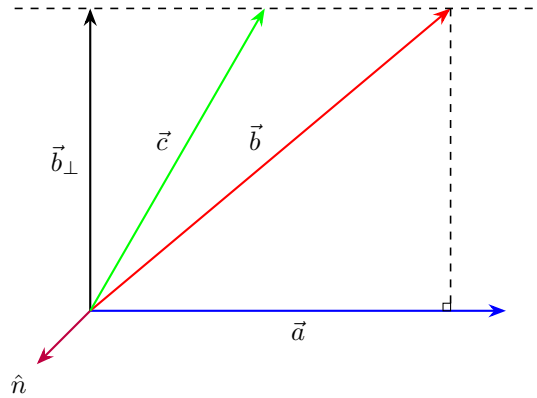
Note that the vector product is not associative with respect to another vector product, that is,

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

- The vector product is *distributive*, that is,

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

- The vector product of \vec{a} and vectors with identical projections perpendicular to \vec{a} , are equal. Consider the following figure.



Then,

$$\vec{a} \times \vec{b} = \vec{a} \times \vec{b}_\perp$$

and

$$\vec{a} \times \vec{c} = \vec{a} \times \vec{c}_\perp.$$

However, from the figure it follows that $\vec{b}_\perp = \vec{c}_\perp$ and we conclude that

$$\vec{a} \times \vec{c} = \vec{a} \times \vec{b}.$$

(For this special instance only!)

Special cases

- If \vec{a} is *parallel* to \vec{b} (meaning the angle θ between \vec{a} and \vec{b} is equal to 0°), then

$$|\vec{a} \times \vec{b}| = 0$$

and thus

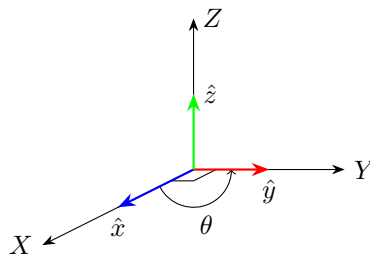
$$\vec{a} \times \vec{b} = \vec{0}.$$

A similar result is true when the angle θ between \vec{a} and \vec{b} is equal to 180° (in this instance, we say that \vec{a} is *antiparallel* to \vec{b}).

- The *vector product of \vec{a} with itself* will yield the *null vector*, that is,

$$\vec{a} \times \vec{a} = (|\vec{a}||\vec{a}| \sin 0^\circ) \hat{n} = \vec{0}.$$

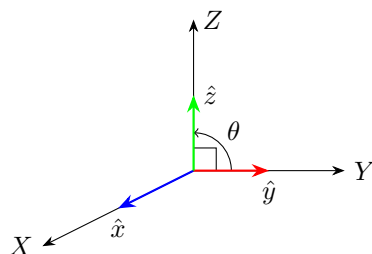
- Let us now calculate the vector product of the Cartesian unit vectors. Consider the following figure.



Note that $\theta = 90^\circ$. It follows that

$$\hat{x} \times \hat{y} = \hat{z}.$$

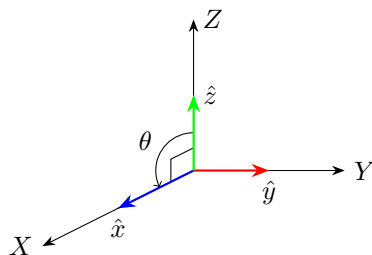
Consider now the following figure.



Again $\theta = 90^\circ$ and we obtain

$$\hat{y} \times \hat{z} = \hat{x}.$$

Lastly, consider the following figure.



Again $\theta = 90^\circ$ and we obtain

$$\hat{z} \times \hat{x} = \hat{y}.$$

Considering the anticommutativity property of the vector product, we note that

$$\hat{y} \times \hat{x} = -\hat{z}, \quad \hat{z} \times \hat{y} = -\hat{x}, \quad \hat{x} \times \hat{z} = -\hat{y}.$$

Finally, we should note that

$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = \vec{0}.$$