Applied Mathematics APM01A1, 2017

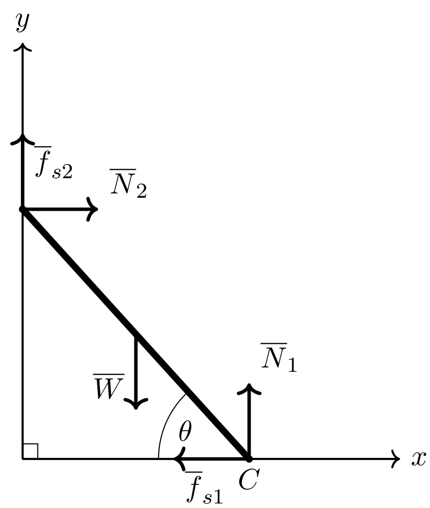
May 11, 2017

Tutorial 9

Question 1

A uniform ladder rests with one end on a horizontal floor and with the other against a vertical wall. The friction coefficients are 2/5 and 1/2 respectively. Calculate the angle of inclination of the ladder if it is about to start sliding.

Solution



First calculate the moment of the object about C. Let the length of the ladder be L. We use the formula that states that the moment is equal to the perpendicular distance from C to the line of action of the force, multiplied by the magnitude of the force. The sign is determined by the direction of rotation: + for anticlockwise rotation and − for clockwise rotation. M_C is

$$M_C = \frac{L}{2}\cos(\theta)\,W - LN_2\sin(\theta) - Lf_{2s}\cos(\theta) = 0.$$

The formula for the friction force $f_{s2} = \mu_2 N_2 = N_2/2$. Then, simplifying the above equation

$$\begin{aligned} \frac{1}{2}\cos(\theta) W - N_2 \sin(\theta) - \frac{N_2}{2}\cos(\theta) &= 0 \\ \frac{W}{2} - \frac{N_2}{2} &= N_2 \tan(\theta) \end{aligned} \tag{1}$$

Now sum the forces in the x and y directions. First, let's sum the forces along the x direction

$$\begin{aligned} N_2 - f_{s1} &= 0 \\ N_2 &= \mu_1 N_1 = \frac{2N_1}{5} \end{aligned} \tag{2}$$

Finally summing the forces in the y direction

$$\begin{aligned} f_{2s} - W + N_1 &= 0 \\ \frac{N_2}{2} - W + N_1 &= 0 \end{aligned}$$

Substituting what we found for N_2 in equation (2) into the above equation

$$\begin{aligned} \frac{N_1}{5} - W + N_1 &= 0 \\ W &= \frac{6N_1}{5} \end{aligned} \tag{3}$$

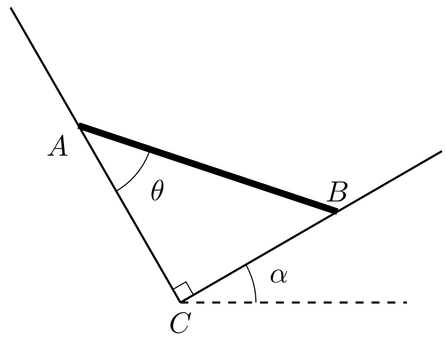
Now substituting into equation (1)

$$\begin{aligned} \frac{6N_1}{10} - \frac{N_1}{5} &= \frac{2N_1}{5}\tan(\theta) \\ \frac{4}{10} &= \frac{2}{5}\tan(\theta) \\ \tan(\theta) = 1 & \end{aligned}$$

So, $\theta = 45°$.

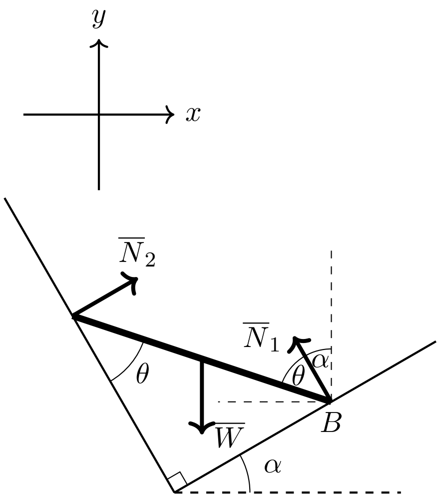
Question 2

In the figure below, a uniform rod AB lies in a vertical plane with its end points against the smooth surfaces CA and CB. Calculate the angle θ for equilibrium if $\alpha = 45°$.



Solution

The free body diagram is



$$
\begin{aligned}
\overline{N}_2 &= N_2 \cos(\alpha)\,\hat{x} + N_2 \sin(\alpha)\,\hat{y} \\
\overline{N}_1 &= -N_1 \sin(\alpha)\,\hat{x} + N_1 \cos(\alpha)\,\hat{y} \\
\overline{W} &= -W\hat{y}.
\end{aligned}
$$

Calculating the moment about point B.

$$
\begin{aligned}
-N_2 L \cos(\theta) + W\frac{L}{2}\cos\left(90^\circ - (\theta+\alpha)\right) &= 0 \\
-N_2 \cos(\theta) + \frac{W}{2}\sin(\theta+\alpha) &= 0 \\
N_2 &= \frac{W}{2}\frac{\sin(\theta+\alpha)}{\cos(\theta)}
\end{aligned}
$$

From summing the x and y direction forces, we obtain

$$
\begin{aligned}
N_1 &= N_2 \cot(\alpha) \\
W &= N_1 \cos(\alpha) + N_2 \sin(\alpha)
\end{aligned}
$$

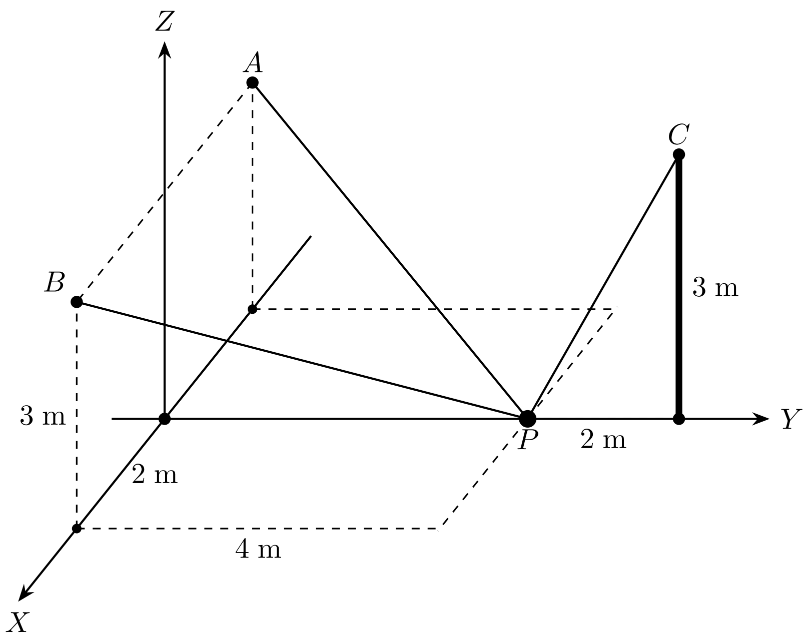
Substituting what we have found into the equation for W, we find

$$
\begin{aligned}
W &= N_2 \cot(\alpha)\cos(\alpha) + N_2 \sin(\alpha) \\
&= N_2 \left(\cot(\alpha)\cos(\alpha) + \sin(\alpha)\right) \\
&= \frac{W}{2}\frac{\sin(\theta+\alpha)}{\cos(\theta)}\left(\cot(\alpha)\cos(\alpha) + \sin(\alpha)\right) \\
\frac{2\cos(\theta)}{\sin(\alpha+\theta)} &= \cot(\alpha)\cos(\alpha) + \sin(\alpha)
\end{aligned}
$$

Solving for θ when $\alpha = 45°$, we find $\theta = 45°$.

Question 3

A weight of 40 N is supported by three cables at point P as shown in the figure below. Calculate the tension in each of the cables



Solution

Labelling the points

$$
\begin{aligned}
A &= (-2,0,3) \\
B &= (2,0,3) \\
C &= (0,6,3) \\
P &= (0,4,0)\,.
\end{aligned}
$$

Next, the directed line segments along with the corresponding unit vectors

$$
\begin{aligned}
\overline{PA} &= -2\hat{x} - 4\hat{y} + 3\hat{z} \\
\frac{\overline{PA}}{|\overline{PA}|} &= -\frac{2}{\sqrt{29}}\hat{x} - \frac{4}{\sqrt{29}}\hat{y} + \frac{3}{\sqrt{29}}\hat{z} \\
\overline{PB} &= 2\hat{x} - 4\hat{y} + 3\hat{z} \\
\frac{\overline{PB}}{|\overline{PB}|} &= \frac{2}{\sqrt{29}}\hat{x} - \frac{4}{\sqrt{29}}\hat{y} + \frac{3}{\sqrt{29}}\hat{z} \\
\overline{PC} &= 2\hat{y} + 3\hat{z} \\
\frac{\overline{PC}}{|\overline{PB}|} &= \frac{2}{\sqrt{13}}\hat{x} + \frac{3}{\sqrt{13}}\hat{z}
\end{aligned}
$$

Thus, the tensions in the cables are

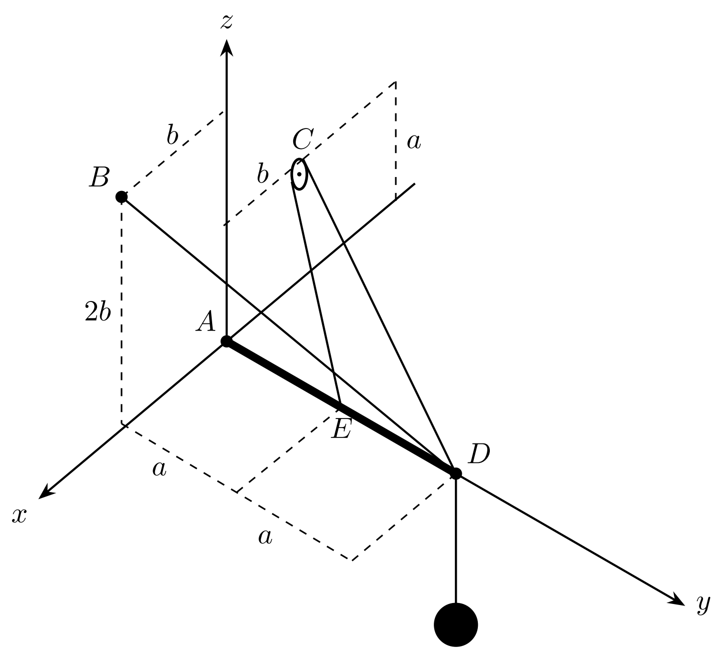
$$
\begin{aligned}
\overline{T}_A &= \frac{T_A}{\sqrt{29}}\left(-2\hat{x} - 4\hat{y} + 3\hat{z}\right)\ \text{N} \\
\overline{T}_B &= \frac{T_B}{\sqrt{29}}\left(2\hat{x} - 4\hat{y} + 3\hat{z}\right)\ \text{N} \\
\overline{T}_C &= \frac{T_C}{\sqrt{13}}\left(2\hat{y} + 3\hat{z}\right)\ \text{N} \\
\overline{W} &= -40\hat{z}\ \text{N}.
\end{aligned}
$$

The equilibrium condition is $\overline{T}_A + \overline{T}_B + \overline{T}_C + \overline{W} = \overline{0}$. This generates three equations, one for the x-component, y-component and one for the z-component. Solving, this gives

$$
\begin{aligned}
T_A &= \frac{20\sqrt{29}}{9}\ \text{N} \\
T_B &= \frac{20\sqrt{29}}{9}\ \text{N} \\
T_C &= \frac{80\sqrt{13}}{9}\ \text{N}.
\end{aligned}
$$

Question 4

A beam AD in the figure below is supported by a ball-and-socket joint at A and two cables, ECD and BD. Cable ECD moves over a frictionless pulley at point C. Calculate the forces in the cables and the reaction at A if a weight of mass W is attached to the beam at D.



Solution

We first calculate the moment of the forces about point A. The forces are

$$\begin{aligned}
\overline{T}_{DB} &= \frac{T_1}{\sqrt{5b^2+4a^2}}\,(b\hat{x} - 2a\hat{y} + 2b\hat{z}) \\
\overline{T}_{DC} &= \frac{T_2}{\sqrt{5a^2+b^2}}\,(-b\hat{x} - 2a\hat{y} + a\hat{z}) \\
\overline{T}_{CE} &= \frac{T_2}{\sqrt{2a^2+b^2}}\,(-b\hat{x} - a\hat{y} + a\hat{z}) \\
\overline{W} &= -W\hat{z}.
\end{aligned}$$

Next, the position vectors from A are

$$\begin{aligned}
\overline{r}_D &= 2a\hat{y} \\
\overline{r}_E &= a\hat{y}.
\end{aligned}$$

Calculating the net moment

$$\begin{aligned}\overline{M}_A &= \overline{r}_D \times \overline{T}_{DB} + \overline{r}_D \times \overline{T}_{DC} + \overline{r}_E \times \overline{T}_{CE} + \overline{r}_D \times \overline{W} \\ &= \left[a\left(\frac{4bT_1}{\sqrt{4a^2+5b^2}} + aT_2\left(\frac{2}{\sqrt{5a^2+b^2}} + \frac{1}{\sqrt{2a^2+b^2}}\right) - 2W\right)\right]\hat{x} \\ &+ \left[ab\left(T_2\left(\frac{2}{\sqrt{5a^2+b^2}} + \frac{1}{\sqrt{2a^2+b^2}}\right) - \frac{2T_1}{\sqrt{4a^2+5b^2}}\right)\right]\hat{z}\end{aligned}$$

Solving for T_1 and T_2, we find

$$T_1 = \frac{W\sqrt{4a^2+5b^2}}{a+2b}, \qquad T_2 = \frac{2W\sqrt{2a^2+b^2}\sqrt{5a^2+b^2}}{(a+2b)\left(2\sqrt{2a^2+b^2}+\sqrt{5a^2+b^2}\right)}$$

Finally, after adding all the forces, including the unknown reaction force $\overline{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$, we find for A_x, A_y and A_z

$$\begin{aligned}A_x &= \frac{bW\sqrt{5a^2+b^2}}{(a+2b)\left(2\sqrt{2a^2+b^2}+\sqrt{5a^2+b^2}\right)} \\ A_y &= \frac{4aw}{a+2b} \\ A_z &= -\frac{aW\sqrt{5a^2+b^2}}{(a+2b)\left(2\sqrt{2a^2+b^2}+\sqrt{5a^2+b^2}\right)}.\end{aligned}$$