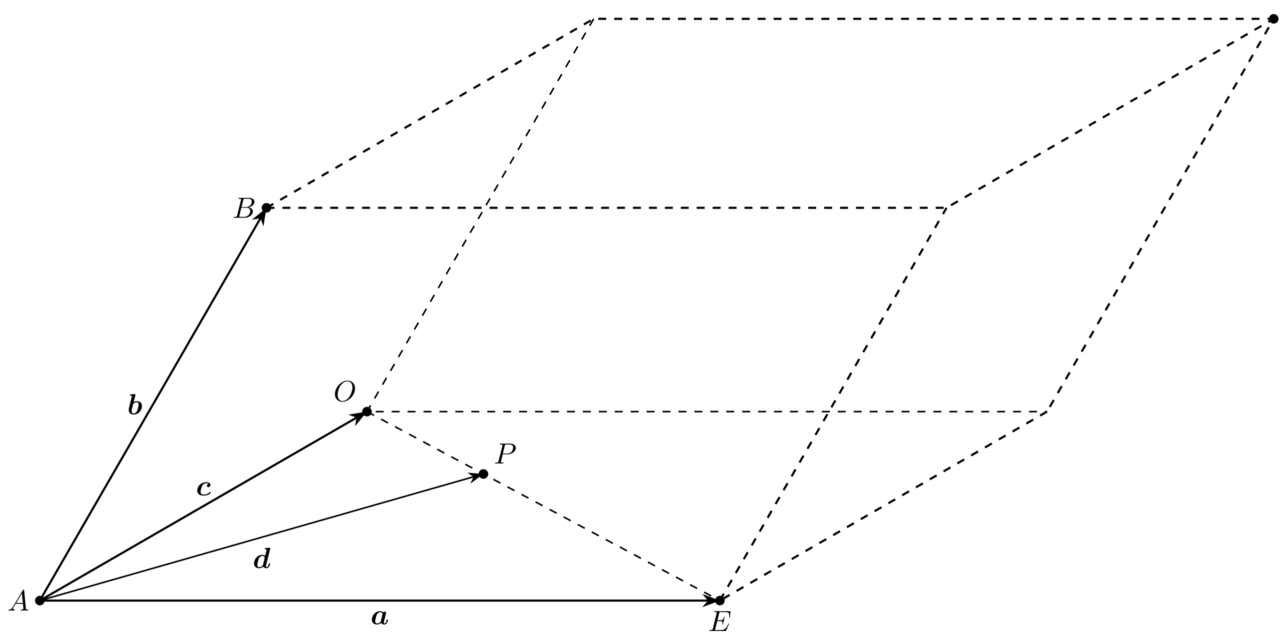
# Applied Mathematics APM01A1, 2017

April 6, 2017

## Tutorial 7

### Question 1

Consider the parallelepiped below.




Given that $\boldsymbol{a} = 2\hat{x} - 4\hat{y} + \hat{z}, \boldsymbol{b} = 2\hat{x} + 3\hat{y} + 7\hat{z}, \boldsymbol{d} = \hat{x} - 2\hat{y} - \hat{z}$ and that $OP : PE = 1 : 2$, calculate the volume of the parallelepiped with sides, $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

### Solution

First note that

$$\overline{OP} = \frac{1}{3}\overline{OE}.$$

From the figure, $\overline{OP} = \boldsymbol{d} - \boldsymbol{c}$ and $\overline{OE} = \boldsymbol{a} - \boldsymbol{c}$. Substituting these expressions for $\overline{OP}$ and $\overline{OE}$, we have

$$\boldsymbol{d} - \boldsymbol{c} = \frac{1}{3}\left(\boldsymbol{a} - \boldsymbol{c}\right).$$

Solving for vector $\boldsymbol{c}$, we find

$$\boldsymbol{c} = \frac{3}{2}\boldsymbol{d} - \frac{1}{2}\boldsymbol{a} = \frac{1}{2}\hat{x} - \hat{y} - 2\hat{z}$$

Now that we have vector $\boldsymbol{c}$, we can calculate the volume from the formula

$$\begin{aligned} V &= \left|\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c})\right| \\ \text{where } \boldsymbol{a} \times \boldsymbol{c} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -4 & 1 \\ 1/2 & -1 & -2 \end{vmatrix} = 9\hat{x} + \frac{9}{2}\hat{y} \end{aligned}$$

Then

$$\begin{aligned} V &= \left|(2\hat{x} + 3\hat{y} + 7\hat{z}) \cdot \left(9\hat{x} + \frac{9}{2}\hat{y}\right)\right| \\ &= \frac{63}{2}\ \mathrm{m}^3. \end{aligned}$$

Note that any permutation of vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in the above formula give the same answer.

## Question 2

Find the equation of the plane that passes through the points $P = (1, 2, 3)\,, Q = (1 - 2, 4)$ and $R = (-2, 5, 3)$.

### Solution

Let $S = (x, y, z)$ be some other point on the plane. Then form the following vectors

$$\begin{aligned} \overline{PS} &= (x-1)\,\hat{x} + (y-2)\,\hat{y} + (z-3)\,\hat{z} \\ \overline{PR} &= -3\hat{x} + 3\hat{y} \\ \overline{PQ} &= -4\hat{y} + \hat{y} \end{aligned}$$

Now calculate $\overline{PS} \cdot \left(\overline{PR} \times \overline{PQ}\right)$

$$\begin{aligned} \overline{PS} \cdot \left(\overline{PR} \times \overline{PQ}\right) &= \left[(x-1)\,\hat{x} + (y-2)\,\hat{y} + (z-3)\,\hat{z}\right] \cdot (3\hat{x} + 3\hat{y} + 12\hat{z}) \\ &= 3(x + y + 4z - 15). \end{aligned}$$

Thus, the equation for the plane is

$$45 = 3(x + y + 4z).$$

Note: different choices of vectors above always lead to the same answer. For example, choosing vectors $\overline{RS}, \overline{RP}$ and $\overline{RQ}$, and then calculating $\overline{RS} \cdot \left(\overline{RP} \times \overline{RQ}\right)$, gives

$$\overline{RS} \cdot \left(\overline{RP} \times \overline{RQ}\right) = -3(x + y + 4z - 15),$$

and the same equation for the plane.

## Question 3

Calculate the shortest distance between the origin and points $P = (3, -2, -1)$, $Q = (1, 3, 4)$ and $R = (2, 1, -2)$.
Hint: First find the unit vector perpendicular to the plane using vectors

$$\begin{aligned} \overline{QP} &= 2\hat{x} - 5\hat{x} - 5\hat{z} \\ \overline{QR} &= \hat{x} - 2\hat{y} - 6\hat{z}. \end{aligned}$$

### Solution

First find the unit vector perpendicular to the plane. As defined above,

$$\begin{aligned} \overline{QP} &= 2\hat{x} - 5\hat{x} - 5\hat{z} \\ \overline{QR} &= \hat{x} - 2\hat{y} - 6\hat{z}. \end{aligned}$$

Then

$$\overline{a} = \overline{QP} \times \overline{QR} = 20\hat{x} + 7\hat{y} + \hat{z}.$$

Now, finding the unit vector

$$\hat{a} = \frac{2\sqrt{2}}{3}\hat{x} + \frac{7}{15\sqrt{2}}\hat{y} + \frac{1}{15\sqrt{2}}\hat{z}.$$

Next, let $\overline{b}$ be the position vector associated to point $Q$: $\overline{b} = \hat{x} + 3\hat{y} + 4\hat{z}$. Calculate $|\overline{b} \cdot \hat{a}|$ to find

$$|\overline{b} \cdot \hat{a}| = \frac{3}{\sqrt{2}} \text{ m}.$$