Applied Mathematics APM01A1, 2017

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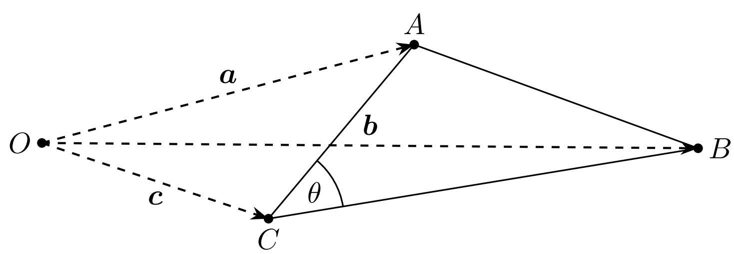
Tutorial 6

Question 1

The vertices of a triangle have $\overline{a}, \overline{b}$ and $\overline{c}$ as position vectors. Show that the angle θ at the vertex with position vector $\overline{c}$ is given by

$$\cos(\theta) = \frac{(\overline{a}-\overline{c})\cdot(\overline{b}-\overline{c})}{|(\overline{a}-\overline{c})\cdot(\overline{a}-\overline{c})(\overline{b}-\overline{c})\cdot(\overline{b}-\overline{c})|^{1/2}}$$

Solution



We are looking for the angle between line segments $\overline{CB}$ and $\overline{CA}$. From the figure,

$$\overline{CB} = \overline{b} - \overline{c}, \qquad \overline{CA} = \overline{a} - \overline{c}.$$

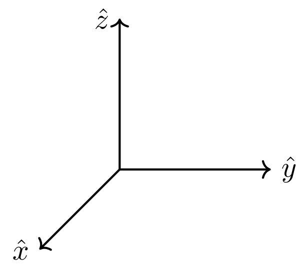
We use the scalar product between $\overline{CA}$ and $\overline{CB}$ to obtain the angle θ:

$$\begin{aligned}(\overline{a}-\overline{c})\cdot(\overline{b}-\overline{c}) &= |\overline{b}-\overline{c}|\,|\overline{a}-\overline{c}|\,\cos(\theta)\\ &= \sqrt{|\overline{b}-\overline{c}|^2\,|\overline{a}-\overline{c}|^2}\cos(\theta)\\ &= \sqrt{(\overline{b}-\overline{c})\cdot(\overline{b}-\overline{c})\,(\overline{a}-\overline{c})\cdot(\overline{a}-\overline{c})}\,\cos(\theta)\end{aligned}$$

Solving for $\cos(\theta)$, we obtain

$$\cos(\theta) = \frac{(\overline{a}-\overline{c})\cdot(\overline{b}-\overline{c})}{\sqrt{(\overline{b}-\overline{c})\cdot(\overline{b}-\overline{c})\,(\overline{a}-\overline{c})\cdot(\overline{a}-\overline{c})}}$$

Question 2



Use the figure above and the right-hand rule to evaluate the following

2.a) $\hat{x} \times \hat{y}$

2.b) $\hat{y} \times \hat{x}$

2.c) $\hat{y} \times (-\hat{x})$

2.d) $(-\hat{x}) \times \hat{y}$

2.e) $\hat{x} \times \hat{z}$

2.f) $\hat{z} \times \hat{y}$

Solution

2.a) $\hat{x} \times \hat{y} = \hat{z}$

2.b) $\hat{y} \times \hat{x} = -\hat{z}$

2.c) $\hat{y} \times (-\hat{x}) = \hat{z}$

2.d) $(-\hat{x}) \times \hat{y} = -\hat{z}$

2.e) $\hat{x} \times \hat{z} = -\hat{y}$

2.f) $\hat{z} \times \hat{y} = -\hat{x}$.

Question 3

Find the area of the triangle which has vertices at the points $(1, 3, 2)$, $(2, -1, 1)$ and $(-1, 2, 3)$ using the vector product.

Solution

Let

$$\begin{aligned} A &= (1,3,2) \\ B &= (2,-1,1) \\ C &= (-1,2,3). \end{aligned}$$

Then form the vectors

$$\begin{aligned} \overline{a} &= \overline{r}_A - \overline{r}_B = -\hat{x} + 4\hat{y} + \hat{z} \\ \overline{b} &= \overline{r}_C - \overline{r}_B = -3\hat{x} + 3\hat{y} + 2\hat{z} \end{aligned}$$

Taking the vector product $\overline{a} \times \overline{b}$, we find

$$\overline{a} \times \overline{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 4 & 1 \\ -3 & 3 & 2 \end{vmatrix} = 5\hat{x} - 1\hat{y} + 9\hat{z}.$$

To obtain the area, we calculate

$$A = \frac{1}{2}|\overline{a} \times \overline{b}| = \frac{\sqrt{107}}{2}\,\mathrm{m}^2 \approx 5.172\,\mathrm{m}^2$$

Question 4

Consider two vectors $\overline{a} = 6\hat{x}$ and $\overline{b} = \cos(\theta)\hat{x} + \sin(\theta)\hat{y}$, where θ is measured counterclockwise from the positive x direction.

4.a) Is $\overline{b}$ a unit vector. Justify your answer?

4.b) Let $0 < \cos(\theta)$ and $0 < \sin(\theta)$. In what direction does $\overline{a} \times \overline{b}$ point?

4.c) Let $\cos(\theta) > 0$ and $0 > \sin(\theta)$. In what direction does $\overline{a} \times \overline{b}$ point?

Solution

4.a) Vector $\overline{b}$ is indeed a unit vector. Calculating its magnitude, we find $|\overline{b}| = 1$.

4.b) $\overline{a} \times \overline{b}$ points in the positive z direction.

4.c) $\overline{a} \times \overline{b}$ points in the negative z direction.

Question 5

Show that

$$\left(\overline{a} \times \overline{b}\right) \cdot \left(\overline{a} \times \overline{b}\right) = \left(\overline{a} \cdot \overline{a}\right)\left(\overline{b} \cdot \overline{b}\right) - \left(\overline{a} \cdot \overline{b}\right)^2$$

Solution

$$\begin{aligned}(\overline{a} \times \overline{b}) \cdot (\overline{a} \times \overline{b}) &= \big(ab\sin(\theta)\big)^2 = a^2b^2\sin^2(\theta) \\ &= a^2b^2\big(1-\cos^2(\theta)\big) \\ &= a^2b^2 - a^2b^2\cos^2(\theta) \\ &= (\overline{a}\cdot\overline{a})(\overline{b}\cdot\overline{b}) - (ab\cos(\theta))^2 \\ &= (\overline{a}\cdot\overline{a})(\overline{b}\cdot\overline{b}) - (\overline{a}\cdot\overline{b})^2\end{aligned}$$

Question 6

Let $\overline{a} = 3\hat{x} + 2\hat{y} + \hat{z}, \overline{b} = \hat{x} - 2\hat{y} + \hat{z}$ and $\overline{c} = \hat{x} - \hat{y} + 2\hat{z}$.

6.a) Calculate $\big(\overline{a} \times \overline{b}\big) \times \overline{c}$

6.b) Is vector $\overline{a}$ perpendicular to $\big(\overline{a} \times \overline{b}\big) \times \overline{c}$? Justify your answer by performing an appropriate calculation.

6.c) Is vector $\overline{c}$ perpendicular to $\big(\overline{a} \times \overline{b}\big) \times \overline{c}$? Justify your answer by performing an appropriate calculation.

Solution

6.a) $\overline{d} = (\overline{a} \times \overline{b}) \times \overline{c} = -12\hat{x} - 16\hat{y} - 2\hat{z}$.

6.b) No, there is no reason why $\overline{a}$ should be perpendicular to $\overline{d}$, and taking the scalar product gives $\overline{a} \cdot \overline{d} = -70$.

6.c) Yes, vector $\overline{c}$ should indeed be perpendicular to $\overline{d}$. Taking the scalar product indeed gives zero $\overline{c} \cdot \overline{d} = 0$.