Applied Mathematics APM01A1, 2017

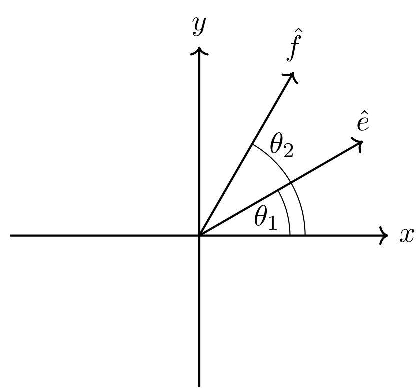
May 19, 2017

Tutorial 10

Question 1

Prove the identity $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ using the scalar product between two unit vectors in the XY-plane.

Solution



Let the two unit vectors be

$$\begin{aligned}\hat{e} &= \cos(\theta_1)\,\hat{x} + \sin(\theta_1)\,\hat{y} \\ \hat{f} &= \cos(\theta_2)\,\hat{x} + \sin(\theta_2)\,\hat{y}\end{aligned}$$

Then take the scalar product between the two vectors

$$\hat{e}\cdot\hat{f} = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2).$$

But, according to the definition of the scalar product,

$$\hat{e}\cdot\hat{f} = \cos(\theta_2-\theta_1)$$

where $\theta_2-\theta_1$ is the angle between the two vectors. Equating the two equations, we find

$$\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) = \cos(\theta_2-\theta_1).$$

Question 2

Referring to figure (1), calculate the resultant force and the resultant moment of the two couples.

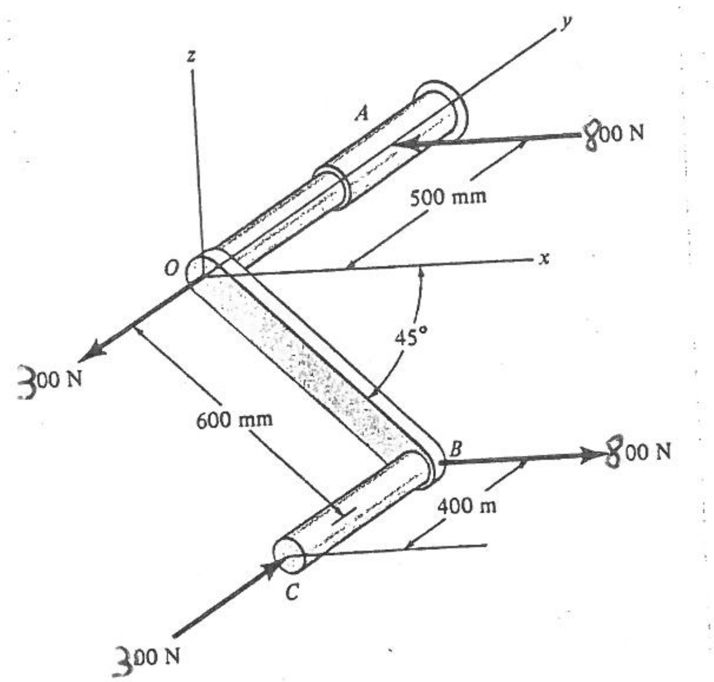


Figure 1:

Solution

The resultant force is zero. The resultant moment is obtained by summing the two moments generated by the two couples. For the $\{800, -800\}$ couple, we have

$$\overline{M}_{800\text{N}} = \overline{r}_{AB} \times (800)\,\hat{x}$$

where $\overline{r}_{AB}$ is the position vector from point A to point B. The coordinates of the two points are

$$A = (0, 0.5, 0)\ \text{m}, \qquad B = \left(0.3\sqrt{2}, 0, -0.3\sqrt{2}\right)\ \text{m}.$$

Thus,

$$\bar{r}_{AB} = 0.3\sqrt{2}\hat{x} - 0.5\hat{y} - 0.3\sqrt{2}\hat{z}.$$

Calculating the vector product gives

$$\overline{M}_{800\text{N}} = \left(0.3\sqrt{2}\hat{x} - 0.5\hat{y} - 0.3\sqrt{2}\hat{z}\right) \times (800\hat{x}) = 400\hat{z} - 240\sqrt{2}\hat{y}.$$

Similarly, $\bar{r}_{OC} = 0.3\sqrt{2}\hat{x} - 0.4\hat{y} - 0.3\sqrt{2}\hat{z}$, and

$$\overline{M}_{300\text{N}} = \left(0.3\sqrt{2}\hat{x} - 0.4\hat{y} - 0.3\sqrt{2}\hat{z}\right) \times (300\hat{y}) = 90\sqrt{2}\hat{z} + 90\sqrt{2}\hat{x}.$$

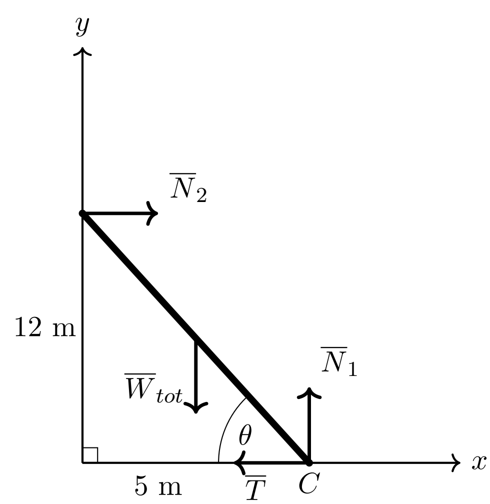
Adding the two together

$$\overline{M}_{res} = \overline{M}_{800\text{N}} + \overline{M}_{300\text{N}} = 127.279\hat{x} - 339.411\hat{y} + 527.279\hat{z}.$$

Question 3

A uniform ladder, length $13a$ and weight W, rests with one end against a smooth wall at a height $12a$ above the smooth horizontal floor and with its other end on the floor. The ladder is prevented from sliding by means of a horizontal rope, the one end of which is tied to the bottom of the ladder and the other end to the wall. A man, weighing $9W$, is climbing the ladder when the rope breaks as he reaches the middle rung. What is the maximum tension the rope can withstand?

Solution



Let's first calculate the moment about point C. There are three forces that contribute. The first is $\overline{N}_2$, the other two are the two weights of W and $9W$ acting at the same point. So $\overline{W}_{tot} = 10W$ N. Thus

$$\overline{M}_C = -N_2\,(12a) + 10W\left(\frac{13}{2}a\right)\left(\frac{5}{13}\right) = 0,$$

where we used the fact that $\cos(\theta) = 5/13$. Solving the above equation, we find

$$N_2 = \frac{25}{12}W.$$

Summing the forces in the x-direction, equating to zero and solving, we find

$$T = N_2 = \frac{25}{12}W.$$

Thus, the tension in the rope when the man is midway up the ladder is roughly $2.08W$ Newtons.

Question 4

Figure (2) is a free body diagram of a uniform rod AB of weight 100N. The rod can rotate freely around A, by means of a pin. However, it is held in equilibrium, at an angle of 15° with the upward vertical, by means of a rope that is tied at B and forms an angle of 30° with the rod and is in the same vertical plane as the rod.

4.a) Calculate the tension T in the rope BC.

4.b) Calculate R, the magnitude of the reaction force, by the pin, at A.

Solution

4.a) Calculating the moment about point A will give the moment due to the tension. Its equation is

$$Tl\sin(30^\circ) - W\frac{l}{2}\sin(15^\circ) = 0,$$

where l is the length of the rod, $l\sin(30^\circ)$ is the perpendicular distance from A to the line of action of the tension and $W = 100$ N. Solving, we find

$$T = 100\sin(15^\circ).$$

4.b) Now, let's sum all of the forces along the x and y directions

$$\begin{aligned} A_x - T\sin(45^\circ) &= 0 \\ A_y - T\cos(45^\circ) - W &= 0, \end{aligned}$$

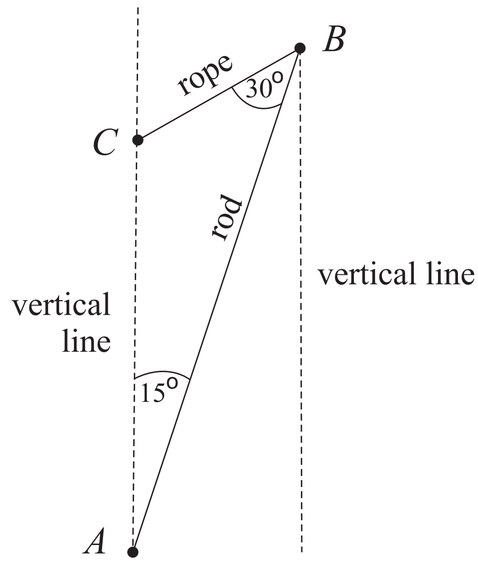


Figure 2:

where 45° is the angle between the rope and vertical line. Solving for the unknowns A_x and A_y

$$A_x = \frac{100}{\sqrt{2}} \sin(15^\circ), \qquad A_y = \frac{100}{\sqrt{2}} \sin(15^\circ) + 100$$

Numerically, the reaction force is

$$\overline{A} = 18.3\hat{x} + 118.3\hat{y}.$$