APPLIED MATHEMATICS

Introduction to Statics
APM01A1/APM1A10

Semester Test 2: 20/04/2017

Duration: 70 minutes **Marks:** 40
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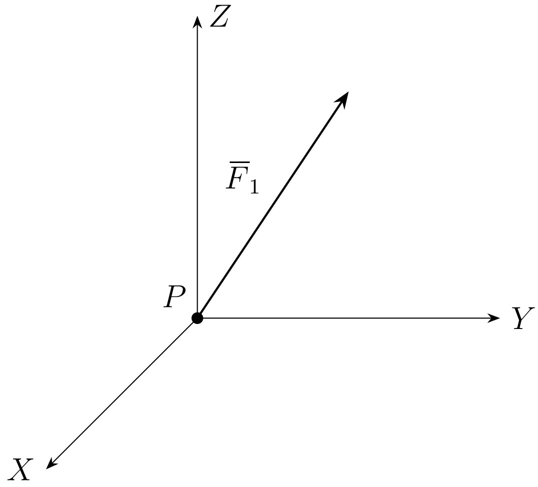
Moderator: Prof M Khumalo

Instructions:

1. Answer all the questions.
2. All calculations must be shown.
3. Only vector algebraic methods may be used.
4. All symbols have their usual meaning.
5. Pocket calculators are permitted.
6. Work to at least three decimal places.
7. All angles are measured in degrees.
8. **No books or notes are allowed.**

Question 1 (10 marks)
Consider the following diagram representing a force $\overline{F}_1$ acting on particle fixed at point P. Let $\overline{F}_1 = (3\hat{x} + 2\hat{y} + 6\hat{z})$ N.



(a) Suppose the mechanism keeping the particle fixed at P is weakest along the direction specified by the position vector $\bar{r} = \hat{x} + 3\hat{y}$. Calculate the component of $\overline{F}_1$ along $\bar{r}$ using the scalar product. (3)

Solution: First, one must find the unit vector specifying the direction of $\bar{r}$:

$$|\bar{r}| = \sqrt{10}, \quad \hat{r} = \frac{\hat{x}}{\sqrt{10}} + \frac{3\hat{y}}{\sqrt{10}}.$$

Then, one computes the scalar product $\overline{F} \cdot \hat{r}$ to find the component of $\overline{F}$ along $\bar{r}$:

$$\begin{aligned}\overline{F}_1 \cdot \hat{r} &= (3\hat{x} + 2\hat{y} + 6\hat{z}) \cdot \left(\frac{\hat{x}}{\sqrt{10}} + \frac{3\hat{y}}{\sqrt{10}}\right) \\ &= \frac{3}{\sqrt{10}} + \frac{6}{\sqrt{10}} \\ &= \frac{9}{\sqrt{10}} \text{ N}\end{aligned}$$

(b) Now let $\overline{F}_2 = -6\hat{x} + 2\hat{y} + 3\hat{z}$. Recalculate the component of $\overline{F}_2$ along $\bar{r}$. (2)

Solution: One performs the scalar product as above with the new force $\overline{F}_2$ instead. The result is

$$\overline{F}_2 \cdot \hat{r} = 0 \text{ N}.$$

(c) Calculate the angle between $\overline{F}_1$ and $\overline{F}_2$ using the scalar product. (5)

Solution: The angle may be calculated using the scalar product

$$\overline{F}_1 \cdot \overline{F}_2 = F_1 F_2 \cos\theta$$

Calculating the left-hand-side of the above equation

$$\overline{F}_1 \cdot \overline{F}_2 = 4 \text{ N}^2.$$

Now calculate the magnitudes of each vector,

$$F_1 = 7 \text{ N}, \quad F_2 = 7 \text{ N}.$$

Substituting these values into the scalar product equation

$$4 = 7^2 \cos\theta.$$

Solving for the angle θ, we find

$$\theta = \arccos\left(\frac{4}{49}\right) \approx 85.318^\circ.$$

Question 2 (10 marks)

Consider the coordinates $P = (0, 0, 0)$, $Q = (1, 2, 3)$ and $R = (0, 0, 3)$.

(a) Use the vector product to find a unit vector that is perpendicular to the plane spanned by $\overline{PQ}$ and $\overline{PR}$, *and* is pointing in the positive X direction. Denote this unit vector by $\hat{e}$. (6)

Solution: First

$$\overline{PQ} = \hat{x} + 2\hat{y} + 3\hat{z}$$
$$\overline{PR} = 3\hat{z}.$$

Calculating the vector product between $\overline{PQ}$ and $\overline{PR}$ gives a vector perpendicular to the plane formed by these two vectors. For it to point in the positive X direction, we need to compute $\overline{PQ} \times \overline{PR}$. Thus,

$$\begin{aligned}\overline{PQ} \times \overline{PR} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} \hat{x} - \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} \hat{y} + \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \hat{z} \\ &= 6\hat{x} - 3\hat{y}.\end{aligned}$$

Calculating $\overline{PR} \times \overline{PQ}$ will not yield the vector asked for in the question. The unit vector is

$$\hat{e} = \frac{\overline{PQ} \times \overline{PR}}{|\overline{PQ} \times \overline{PR}|} = \frac{6}{\sqrt{45}}\hat{x} - \frac{3}{\sqrt{45}}\hat{y}.$$

(b) Calculate the area of the triangle having vertices P, Q and R. (2)

Solution: The area of the triangle is found to be

$$A = \frac{1}{2}|\overline{PQ} \times \overline{PR}| = \frac{1}{2}|6\hat{x} - 3\hat{y}| = \frac{\sqrt{45}}{2}.$$

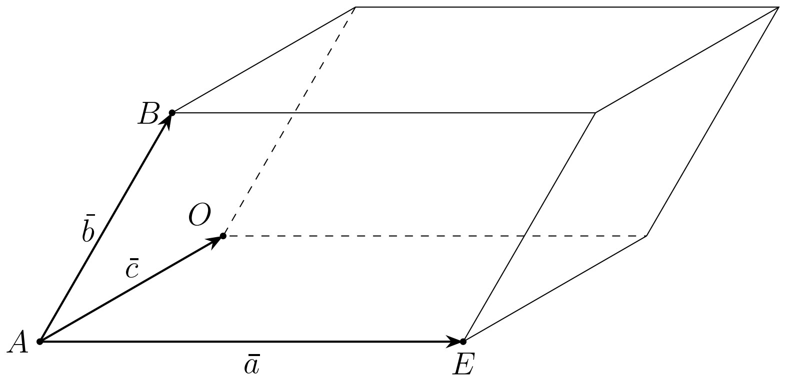
(c) Calculate the area of the parallelogram with sides $\overline{PQ}$ and $\overline{PR}$. (2)

Solution: The area of the parallelogram is found to be

$$A = |\overline{PQ} \times \overline{PR}| = \sqrt{45}.$$

Question 3 (10 marks)

(a) Consider the parallelepiped below. Let $\bar{a} = -3\hat{x} + 3\hat{y}$, $\bar{b} = 2\hat{x} + 4\hat{y} + 10\hat{z}$ and $\bar{c} = 12\hat{x}$. Calculate the volume of this parallelepiped. (6)



Solution: The formula for the volume is

$$V = |\bar{b} \cdot (\bar{a} \times \bar{c})|.$$

Calculating the vector product first, we find:

$$\bar{a} \times \bar{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 12 & 0 & 0 \\ -3 & 3 & 0 \end{vmatrix} = -36\hat{z}.$$

Now take the scalar product with vector $\bar{b}$:

$$\bar{b} \cdot (-36\hat{z}) = (2\hat{x} + 4\hat{y} + 10\hat{z}) \cdot (-36\hat{z}) = -360.$$

The volume formula requires us to take the absolute value and thus

$$V = 360.$$

(b) State the reason that the order of the vectors $\bar{a}, \bar{b}$ and $\bar{c}$ in the parallelepiped volume formula does not influence the final result. (2)

Solution: The result of the scalar triple product is always the same up to a sign. Since we take the absolute value of the scalar triple product, the sign, whether positive or negative, becomes positive.

(c) Suppose the result of calculating the scalar triple product between vectors $\bar{r}_1, \bar{r}_2$ and $\bar{r}_3$ is: (2)

$$\bar{r}_1 \cdot (\bar{r}_2 \times \bar{r}_3) = 5.$$

Calculate $\bar{r}_3 \cdot (\bar{r}_2 \times \bar{r}_1)$ and $\bar{r}_3 \cdot (\bar{r}_1 \times \bar{r}_2)$.

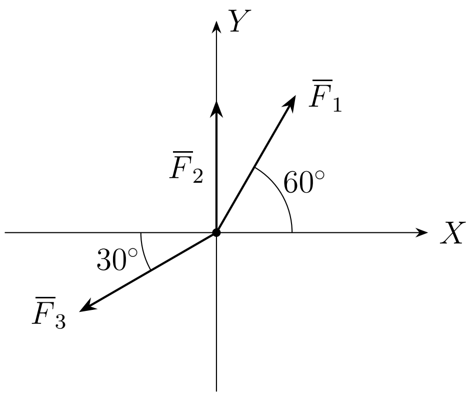
Solution: In the first calculation, we only need to notice that a single swap has been performed from the given product - that of vectors $\bar{r}_1$ and $\bar{r}_3$. This swap generates an extra minus sign only:

$$\bar{r}_3 \cdot (\bar{r}_2 \times \bar{r}_1) = -5.$$

In the second calculation, two swaps have been performed generating two minus signs. Thus,

$$\bar{r}_3 \cdot (\bar{r}_1 \times \bar{r}_2) = 5.$$

Question 4 (10 marks)
In the figure below, let $F_1 = 4$ N, $F_2 = 2\sqrt{3}$ N, and $F_3 = 5$ N.



(a) Calculate the resultant force acting on the particle. (5)

Solution: To calculate the resultant force, one must sum up all the forces acting upon the particle,

$$\overline{F}_{\text{res}} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3.$$

Writing each force in component form:

$$\overline{F}_1 = \left(4\cos(60^\circ)\,\hat{x} + 4\sin(60^\circ)\,\hat{y}\right) \text{ N} = \left(2\hat{x} + 2\sqrt{3}\hat{y}\right) \text{ N},$$
$$\overline{F}_2 = \left(2\sqrt{3}\hat{y}\right) \text{ N},$$
$$\overline{F}_3 = \left(-5\cos(30^\circ)\,\hat{x} - 5\sin(30^\circ)\,\hat{y}\right) \text{ N} = \left(-\frac{5\sqrt{3}}{2}\hat{x} - \frac{5}{2}\hat{y}\right) \text{ N}.$$

Adding up these vectors is done component-wise:

$$\begin{aligned}\overline{F}_{\text{res}} &= \left(\left(2 - \frac{5\sqrt{3}}{2}\right)\hat{x} + \left(2\sqrt{3} + 2\sqrt{3} - \frac{5}{2}\right)\hat{y}\right)\ \text{N} \\ &= \left(\left(\frac{4 - 5\sqrt{3}}{2}\right)\hat{x} + \left(\frac{8\sqrt{3} - 5}{2}\right)\hat{y}\right)\ \text{N} \\ &\approx (-2.3301\hat{x} + 4.4282\hat{y})\ \text{N}\end{aligned}$$

(b) Is the system in equilibrium or not? (2)

Solution: The system is not in equilibrium.

(c) Determine the force that would need to be added to this system for it to be in equilibrium. (3)

Solution: We have already calculated

$$\overline{F}_1 + \overline{F}_2 + \overline{F}_3 = -2.330\hat{x} + 4.428\hat{y}.$$

Manipulating this equation

$$\overline{F}_1 + \overline{F}_2 + \overline{F}_3 + 2.330\hat{x} - 4.428\hat{y} = 0\hat{x} + 0\hat{y}.$$

Thus, the force needed to bring the system into the equilibrium is

$$\overline{F} = (2.330\hat{x} - 4.428\hat{y})\ \text{N}.$$