APPLIED MATHEMATICS

Introduction to Statics
APM01A1/APM1A10

Semester Test 1: 16/03/2017

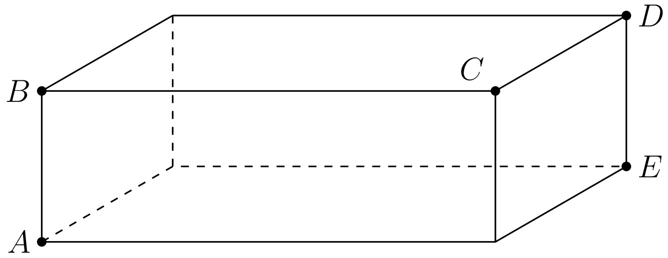
Duration: 60 minutes **Marks:** 40
Assessor: Mr KD Anderson, Dr GJ Kemp
Moderator: Prof M Khumalo

Instructions:

1. Answer all the questions.
2. Each new question must start on a new page.
3. All calculations must be shown.
4. All symbols have their usual meaning.
5. Pocket calculators are permitted.
6. Work to at least three decimal places.
7. All angles measured in degrees.
8. **No books or notes are allowed.**

Question 1 (11 marks)

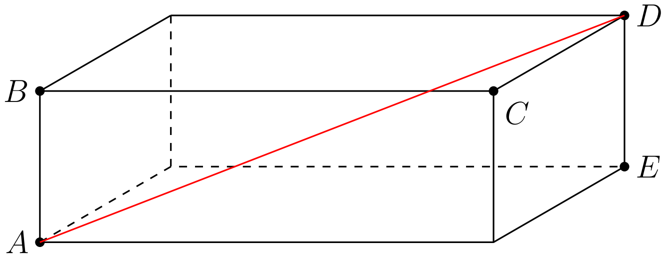
Consider the following figure for reference:



A helicopter flies 200 m vertically upwards from A to B, then 600 m horizontally North to C and then 400 m horizontally West to D.

(a) Calculate the total distance the helicopter travelled. (2)

Solution:

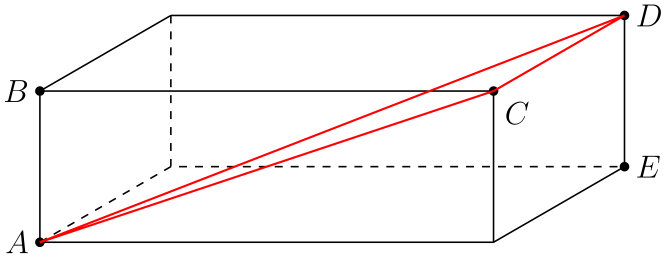


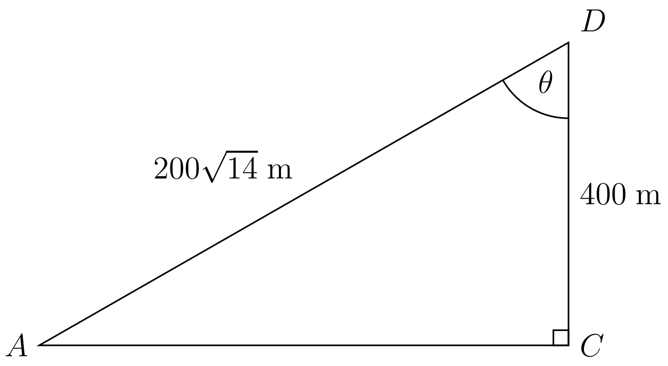
The total distance is given by the magnitude of $\overline{AD}$:

$$|\overline{AD}| = \sqrt{(200)^2 + (600)^2 + (400)^2} = 200\sqrt{14} \text{ m} = 748,331 \text{ m}.$$

(b) Calculate $\angle ADC$. (3)

Solution: Consider the right-angled triangle ACD in the following figure:





From the latter figure it follows that

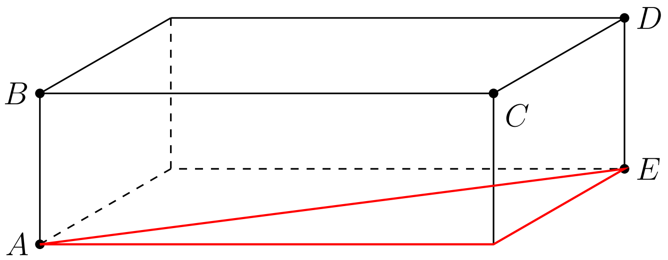
$$\cos\theta = \frac{400}{200\sqrt{14}}$$

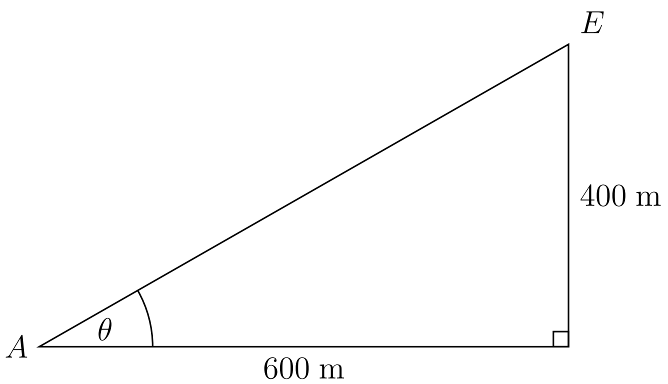
and thus

$$\theta \approx 57,688°.$$

(c) Calculate the cardinal direction $\overline{AD}$ in Degrees West of North. (3)

Solution: The cardinal direction is found by identifying the triangle in the following figure:





From the latter figure it follows that

$$\tan\theta = \frac{400}{600}$$

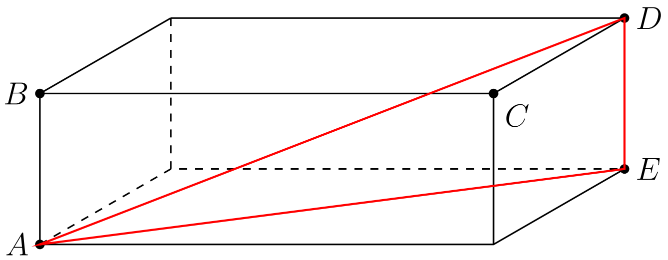
and thus

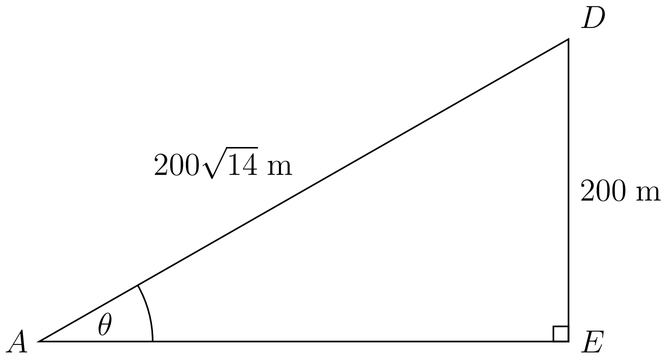
$$\theta \approx 33,690°.$$

We conclude that the point E is $33,690°$ West of North.

(d) Calculate the angle of elevation of $\overline{AD}$. (3)

Solution: For the angle of elevation





From the latter figure it follows that

$$\sin\theta = \frac{200}{200\sqrt{14}}$$

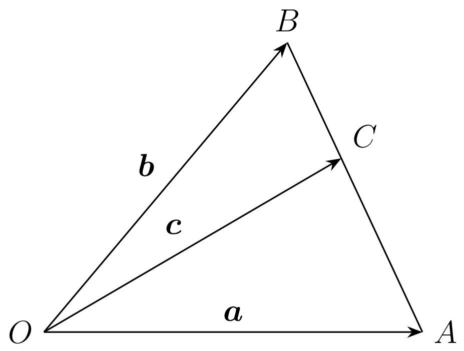
from which we obtain

$$\theta \approx 15,501°.$$

The angle of elevation is therefore $15,501°$ measured from the horizontal plane.

Question 2 (8 marks)
Consider any triangle $\triangle OAB$ with a point C on AB. Let $\boldsymbol{a} = \overline{OA}$, $\boldsymbol{b} = \overline{OB}$ and $\boldsymbol{c} = \overline{OC}$. It follows that $\boldsymbol{c} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}$, where $\lambda + \mu = 1$. Find the values of λ and μ if C is between A and B with $\left|\overline{AC}\right| : \left|\overline{CB}\right| = 1 : 2$

Solution: Consider the following figure:



From the ratio $\left|\overline{AC}\right| : \left|\overline{CB}\right| = 1 : 2$, we conclude that

$$\frac{\overline{AC}}{\overline{CB}} = \frac{1}{2}$$

which implies that

$$\overline{AC} = \frac{1}{2}\overline{CB}$$

and equally that $\overline{CB} = 2\overline{AC}$. Since $\overline{AB} = \overline{AC} + \overline{CB}$, and using the formula above, it follows that

$$\overline{AB} = \overline{AC} + \overline{CB} = \overline{AC} + 2\overline{AC} = 3\overline{AC}$$

and thus we conclude that

$$\overline{AC} = \frac{1}{3}\overline{AB}.$$

Writing $\overline{AC}$ and $\overline{AB}$ in terms of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ yields

$$\overline{AC} = \boldsymbol{a} - \boldsymbol{c},$$
$$\overline{AB} = \boldsymbol{a} - \boldsymbol{b}.$$

We can now substitute and rearrange to find

$$\boldsymbol{c} = \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}.$$

Substituting for $\boldsymbol{c}$, we find

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} = \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b},$$

and thus

$$\lambda = \frac{2}{3}, \quad \mu = \frac{1}{3}.$$

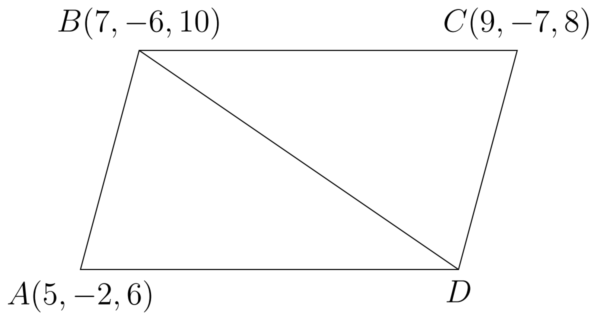
Question 3 (13 marks)

Consider the parallelogram $ABCD$ where the coordinates for points A, B and C are

$(5, -2, 6)$, $(7, -6, 10)$ and $(9, -7, 8)$ respectively.

(a) Calculate $\overline{BD}$ in component form. (3)

Solution: Consider the following figure:



It follows that

$$\overline{BC} = 2\hat{x} - \hat{y} - 2\hat{z},$$
$$\overline{AB} = 2\hat{x} - 4\hat{y} + 4\hat{z}. \qquad (\text{equivalently } \overline{BA} = -2\hat{x} + 4\hat{y} - 4\hat{z})$$

Then

$$\begin{aligned}\overline{BD} &= \overline{BC} + \overline{CD} = \overline{BC} + \overline{BA}\\ &= (2\hat{x} - \hat{y} - 2\hat{z}) + (-2\hat{x} + 4\hat{y} - 4\hat{z})\\ &= 3\hat{y} - 6\hat{z}.\end{aligned}$$

(b) Calculate the unit vector $\hat{e}$ that has the same direction as $\overline{BD}$. (3)

Solution: The unit vector in the same direction of $\overline{BD}$ is given by

$$\hat{e} = \frac{\overline{BD}}{|\overline{BD}|}.$$

Note that

$$|\overline{BD}| = \sqrt{(3)^3 + (6)^2} = \sqrt{45} = 3\sqrt{5},$$

and thus

$$\hat{e} = \frac{3\hat{y} - 6\hat{z}}{3\sqrt{5}} = \frac{1}{\sqrt{5}}\hat{x} - \frac{2}{\sqrt{5}}\hat{y}.$$

(c) Let $\angle CBD = 63^o$. Calculate the component and projection of $\overline{BC}$ in the $\overline{BD}$ direction. (3)

Solution: The magnitude of $\overline{BC}$ is

$$\sqrt{2^2 + (-1)^2 + (-2)^2} = 3.$$

The component of $\overline{BC}$ in the e direction is

$$(BC)_e = 3\cos(63°) \approx 1,362.$$

The projection is given by

$$(\overline{BC})_e = (BC)_e\hat{e} \approx 1,362\hat{e} \approx 0,609\hat{x} - 1,218\hat{y}.$$

(d) Calculate the position vector, $\boldsymbol{r}_D$, corresponding to point D in component form. (4)

Solution: Let (x, y, z) be the coordinates of D. The coordinates are found from the fact that $ABCD$ is a parallelogram and $\overline{AD} = \overline{BC}$. The component form of vector $\overline{AD}$ is

$$(x-5)\hat{x} + (y-(-2))\hat{y} + (z-6)\hat{z},$$

and the component form of vector $\overline{BC}$ is $2\hat{x} - \hat{y} - 2\hat{z}$. Thus,

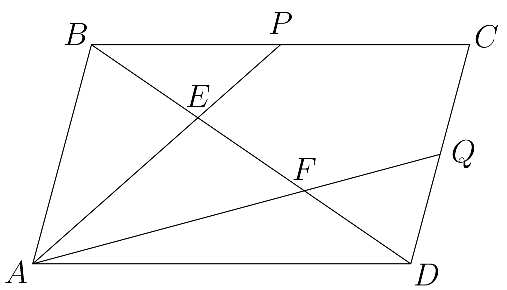
$$\begin{aligned} x - 5 &= 2 &\Rightarrow\quad& x = 7 \\ y - (-2) &= -1 &\Rightarrow\quad& y = -3 \\ z - 6 &= -2 &\Rightarrow\quad& z = 4. \end{aligned}$$

The components of the position vector $\boldsymbol{r}_D$ is given by the coordinates of D and so

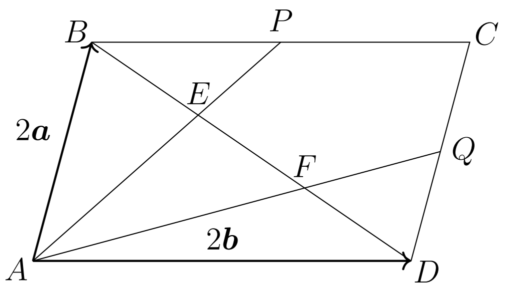
$$\boldsymbol{r}_D = 7\hat{x} - 3\hat{y} + 4\hat{z}.$$

Question 4 (8 marks)

In the following figure, $ABCD$ is a parallelogram with P and Q the midpoints of the sides BC and CD respectively. Show that AP and AQ trisect the diagonal BD at points E and F.



Solution: Let $2\boldsymbol{a} = \overline{AB}$ and $2\boldsymbol{b} = \overline{AD}$ (as illustrated in the following figure).



In $\triangle ABC$, we observe that

$$\overline{PE} = \alpha\overline{AP} \tag{1}$$

and rewriting this in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ yields

$$\overline{BE} - \boldsymbol{b} = \alpha(2\boldsymbol{a} + \boldsymbol{b})$$

and thus

$$\overline{BE} = 2\alpha\boldsymbol{a} + (1 + \alpha)\boldsymbol{b}. \tag{2}$$

Consider now the diagonal $\overline{BD}$ and notice that

$$\overline{BE} = \beta\overline{BD} \tag{3}$$

since B, E and D lie on the same straight line. Note that $\overline{BD} = 2\boldsymbol{b} - 2\boldsymbol{a}$. Substituting this information and equation (2) into equation (1) yields

$$2\alpha\boldsymbol{a} + (1 + \alpha)\boldsymbol{b} = \beta(2\boldsymbol{b} - 2\boldsymbol{a}).$$

By algebraic manipulation of the above expression, we obtain

$$(2\alpha + 2\beta)\boldsymbol{a} = (2\beta - \alpha - 1)\boldsymbol{b}. \tag{4}$$

However, since $\boldsymbol{a} \neq \boldsymbol{b}$, this equation can only be true if we have the null vector on both sides of equation (4), which implies that

$$\begin{aligned} 2\alpha + 2\beta &= 0, \\ 2\beta - \alpha - 1 &= 0. \end{aligned}$$

The first equation above yields $\alpha = \beta$ and substituting it into the second equation above yields

$$\begin{aligned} 2\beta - (-\beta) - 1 &= 0 \\ 2\beta + \beta &= 1 \\ \beta &= \frac{1}{3} \end{aligned}$$

Thus,

$$\overline{BE} = \frac{1}{3}\overline{BD}$$

and the conclusion follows.