APPLIED MATHEMATICS

Introduction to Statics
APM01A1/APM1A10

Supplementary Test 1: 23/03/2017

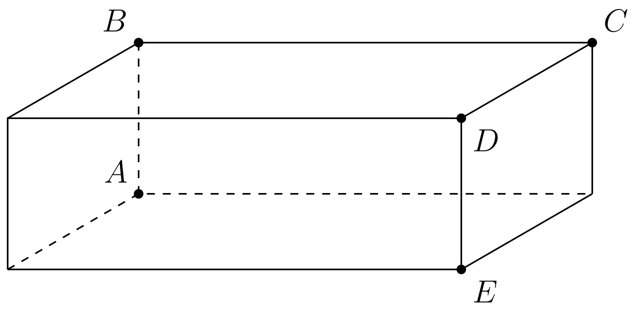
Duration: 70 minutes **Marks:** 40
Assessor: Mr KD Anderson, Dr GJ Kemp
Moderator: Prof M Khumalo

Instructions:

1. Answer all the questions.
2. Each new question must start on a new page.
3. All calculations must be shown.
4. All symbols have their usual meaning.
5. Pocket calculators are permitted.
6. Work to at least three decimal places.
7. All angles measured in degrees.
8. **No books or notes are allowed.**

Question 1 (11 marks)

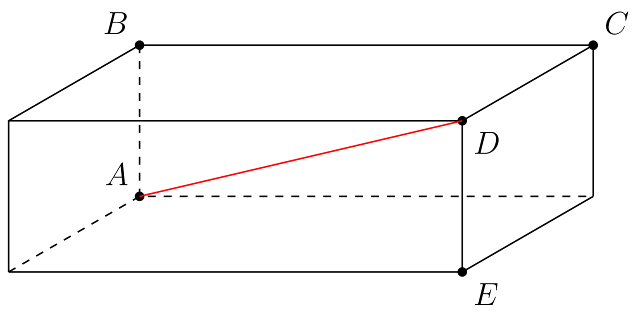
Consider the following figure for reference:



A helicopter flies 200 m vertically upwards from A to B, then 600 m horizontally North to C and then 300 m horizontally East to D.

(a) Calculate the magnitude of the total displacement of the helicopter. (2)

Solution:

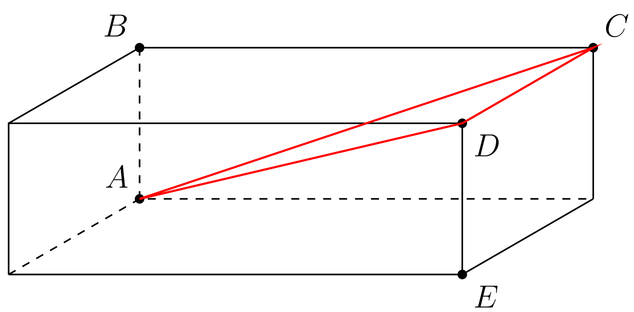


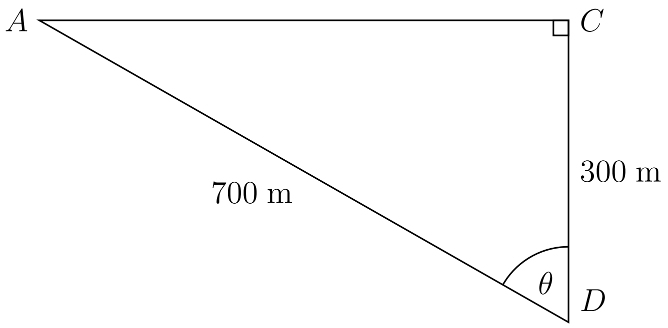
The total displacement is given by the magnitude of $\overline{AD}$:

$$\left|\overline{AD}\right| = \sqrt{(200)^2 + (600)^2 + (300)^2} = 700 \text{ m.}$$

(b) Calculate $\angle ADC$. (3)

Solution: Consider the right-angled triangle ADC in the following figure:





From the latter figure it follows that

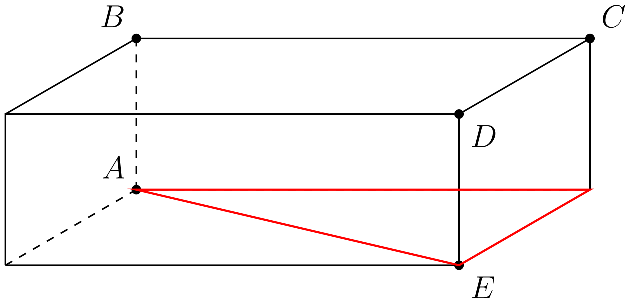
$$\cos\theta = \frac{300}{700}$$

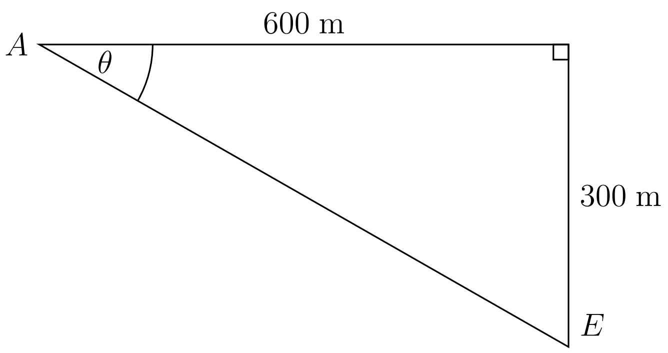
and thus

$$\theta \approx 64,623^\circ.$$

(c) Calculate the cardinal direction $\overline{AD}$ in Degrees East of North. (3)

Solution: The cardinal direction is found by identifying the triangle in the following figure:





From the latter figure it follows that

$$\tan\theta = \frac{300}{600}$$

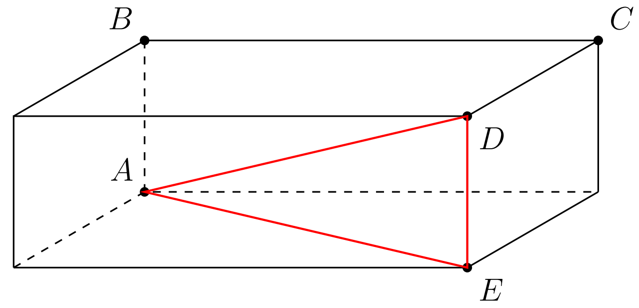
and thus

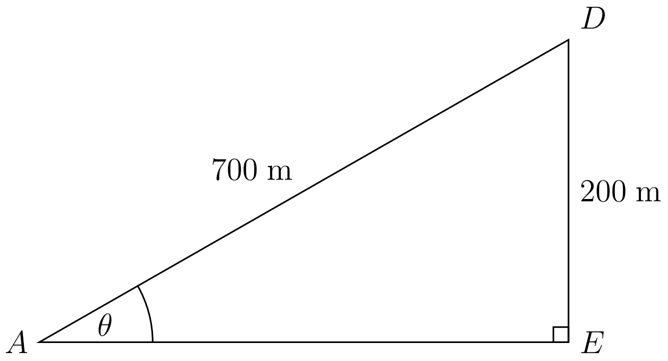
$$\theta \approx 26,565^\circ.$$

We conclude that the point E is $26,565^\circ$ East of North.

(d) Calculate the angle of elevation of $\overline{AD}$. (3)

Solution: For the angle of elevation





From the latter figure it follows that

$$\sin\theta = \frac{200}{700}$$

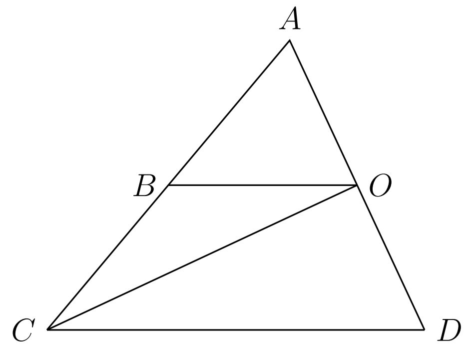
from which we obtain

$$\theta \approx 16{,}601^\circ.$$

The angle of elevation is therefore $16{,}601^\circ$ measured from the horizontal plane.

Question 2 (8 marks)

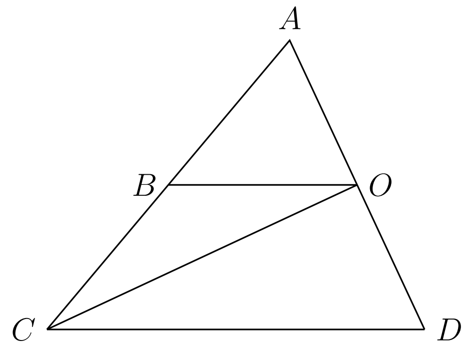
In the figure below,



$\overline{OB}$ is parallel to $\overline{DC}$ and $\left|\overline{OA}\right| : \left|\overline{DO}\right| = 1 : 2$. Which value of k will satisfy

$$k = \frac{\left|\overline{OB}\right|}{\left|\overline{DC}\right|}.$$

Solution: Consider the following figure:



Using the auxiliary result, $\overline{OB}$ can be written as

$$\overline{OB} = \alpha\overline{OA} + (1-\alpha)\overline{OC}.$$

But, from the given ratios we conclude that

$$\overline{OA} = \frac{1}{3}\overline{DA},$$

and hence $\overline{OB}$ becomes

$$\overline{OB} = \frac{\alpha}{3}\overline{DA} + (1-\alpha)\overline{OC}.$$

Next, note that

$$\overline{DC} = \overline{DO} + \overline{OC},$$

and using the given ratio again, we obtain

$$\overline{DC} = \frac{2}{3}\overline{DA} + \overline{OC}.$$

Substituting what we have obtained for $\overline{DC}$ and $\overline{OB}$ into the equation $\overline{OB} = k\overline{DC}$, we find the following:

$$\frac{\alpha}{3}\overline{DA} + (1-\alpha)\overline{OC} = \frac{2k}{3}\overline{DA} + k\overline{OC}.$$

Factorizing and simplifying the above equation, we find

$$\left(\frac{\alpha}{3} - \frac{2k}{3}\right)\overline{DA} = (k+\alpha-1)\overline{OC}.$$

Since $\overline{DA} \neq \overline{OC}$, the above equation is only valid if we have the null vector on both sides, which implies that

$$\frac{\alpha}{3} - \frac{2k}{3} = 0,$$
$$k + \alpha - 1 = 0.$$

Solving for k, we obtain

$$k = \frac{1}{3}.$$

Question 3 (13 marks)

Consider the point P with coordinates $(3, 4, 2)$.

(a) Write the position vector $\boldsymbol{r}$, corresponding to P, in component form. (1)

> **Solution:**
> $$\boldsymbol{r} = 3\hat{x} + 4\hat{y} + 2\hat{z}.$$

(b) Without using the scalar product, calculate each angle between $\boldsymbol{r}$ and the vectors $\boldsymbol{a} = 6\hat{x}$, $\boldsymbol{b} = 7\hat{y}$ and $\boldsymbol{c} = \hat{z}$. (6)

> **Solution:** Since the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are parallel to the X, Y, and Z axes respectively, these angles are simply the angles α, β and γ of the direction cosines.
> $$\cos\alpha = \frac{r_x}{r} = \frac{3}{\sqrt{29}} \implies \alpha = 56.145^\circ$$
> $$\cos\beta = \frac{r_y}{r} = \frac{4}{\sqrt{29}} \implies \beta = 42.031^\circ$$
> $$\cos\gamma = \frac{r_z}{r} = \frac{2}{\sqrt{29}} \implies \gamma = 68.120^\circ$$

(c) Without using the scalar product, calculate the projection of $\boldsymbol{r}$ in the direction of $\boldsymbol{e} = 6\hat{x} + 8\hat{y}$ in component form. (4)

> **Solution:** The projection of $\boldsymbol{r}$ in the direction of the vector $\boldsymbol{e}$ has the form
> $$\boldsymbol{r}_e = (r\cos\theta)\,\hat{e},$$
> where θ is the angle between $\boldsymbol{r}$ and $\hat{e}$, the unit vector in the same direction as $\boldsymbol{e}$. The unit vector $\hat{e}$ is calculated to be
> $$\hat{e} = \frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}.$$
> The angle θ may be obtained by using γ from part (b):
> $$90^\circ - \gamma = 21.8^\circ.$$
> The magnitude r may also be obtained from part (b):
> $$r = \sqrt{29}\ \text{m}.$$
> Thus, the projection is
> $$\boldsymbol{r}_e = \sqrt{29}\cos(21.8^\circ)\hat{e} = 3\hat{x} + 4\hat{y}.$$

(d) Write the projection you calculated in the previous question as $\boldsymbol{r}_e$, and calculate $|\boldsymbol{r} - \boldsymbol{r}_e|$. (2)

Solution: First, calculate $\boldsymbol{r} - \boldsymbol{r}_e$:

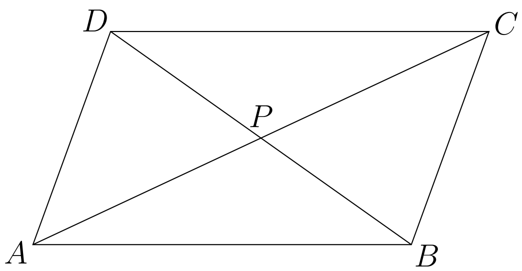
$$\boldsymbol{r} - \boldsymbol{r}_e = 2\hat{z}.$$

Calculating the magnitude of the difference between these two vectors yields

$$|\boldsymbol{r} - \boldsymbol{r}_e| = 2 \text{ m}.$$

Question 4 (8 marks)

Use the figure below to show that the diagonals of a parallelogram bisect each other.



Solution: Let $\overline{DC} = \boldsymbol{a}$ and $\overline{AD} = \boldsymbol{b}$. Consider $\triangle ACD$ in the figure. The points A, P and C lie on a straight line from which follows:

$$\overline{AP} = \alpha\overline{AC}. \tag{1}$$

Consider now the triangle $\triangle ABD$. The points D, P and B are also on a straight line, and thus

$$\overline{BP} = \beta\overline{BD}. \tag{2}$$

Rewriting equation 2 in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ yields

$$\overline{AP} - \boldsymbol{a} = \beta\left(\boldsymbol{b} - \boldsymbol{a}\right) \tag{3}$$

Substitute equation 3 into equation 1 to obtain

$$\beta\boldsymbol{b} + (1-\beta)\boldsymbol{a} = \alpha(\boldsymbol{b} + \boldsymbol{a}). \tag{4}$$

Multiplying out and gathering coefficients of the same vectors in equation 4, we obtain

$$(\beta - \alpha)\boldsymbol{b} = (\alpha + \beta - 1)\boldsymbol{a}.$$

Since $\boldsymbol{a} \neq \boldsymbol{b}$, this equation is only true if the null vector occurs on both sides, and thus

$$\beta - \alpha = 0, \quad \alpha + \beta - 1 = 0.$$

Solving the above equations gives

$$\alpha = \beta = \frac{1}{2}.$$

Hence

$$\overline{AP} = \frac{1}{2}\overline{AC}$$

and

$$\overline{BP} = \frac{1}{2}\overline{BD}$$

which proves that the diagonals of a parallelogram bisect each other.