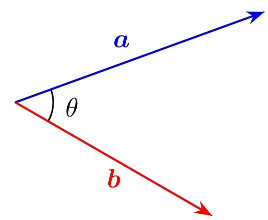
**Scalar product**

- The *scalar product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is yields a *scalar*.
- We denote it by $\boldsymbol{a} \cdot \boldsymbol{b}$.
- It is defined to be the product of the *magnitudes of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ and *the cosine of the smallest angle* $\theta$ *between their forward directions*, that is,

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta. \tag{1}$$




- The scalar product of a vector with itself yields

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}| \cos 0^\circ = |\boldsymbol{a}|^2.$$

Therefore, the scalar product of a vector with itself yields the square of the magnitude of the vector. The scalar product of a vector of itself is also denoted as

$$\boldsymbol{a}^2 \coloneqq \boldsymbol{a} \cdot \boldsymbol{a}. \tag{2}$$

(Note that this is the only time a power is allowed with a vector; for example, $\boldsymbol{a}^3$ or $\boldsymbol{a}^n$ has no meaning.)

- The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to zero if and only if

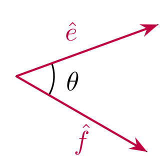
$$\boldsymbol{a} = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{b} = \boldsymbol{0} \qquad \text{or} \qquad \theta = 90^\circ.$$

If $\theta = 90^\circ$, then $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ and we denote this fact by $\boldsymbol{a} \perp \boldsymbol{b}$.

- If $\hat{e}$ and $\hat{f}$ are two unit vectors and $\theta$ is the angle between their forward directions, then

$$\hat{e} \cdot \hat{f} = |\hat{e}||\hat{f}| \cos\theta = \cos\theta$$

(since $|\hat{e}| = 1$ and $|\hat{f}| = 1$).



- It follows for the Cartesian unit vectors that

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

  and

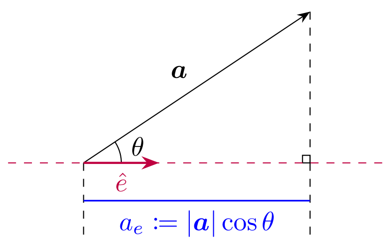
$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

  since, for example,

$$\begin{aligned} \hat{x} \cdot \hat{x} &= |\hat{x}||\hat{x}| \cos 0^\circ = 1, \\ \hat{x} \cdot \hat{y} &= |\hat{x}||\hat{y}| \cos 90^\circ = 0. \end{aligned}$$

- The component of $\boldsymbol{a}$ in the direction of the vector $\hat{e}$ can be determined by using the scalar product as follows:

$$\begin{aligned} \boldsymbol{a} \cdot \hat{e} &= |\boldsymbol{a}||\hat{e}| \cos \theta \\ &= |\boldsymbol{a}| \cos \theta \\ &= a_e. \end{aligned}$$




- The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the Cartesian components of these two vectors is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z,$$

  where

$$\begin{aligned} \boldsymbol{a} &= a_x \hat{x} + a_y \hat{y} + a_z \hat{z}, \\ \boldsymbol{b} &= b_x \hat{x} + b_y \hat{y} + b_z \hat{z}. \end{aligned}$$

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**Algebraic properties**

- The scalar product is *not closed* (this means that the product does not yield a vector).
- The scalar product is *commutative*, that is,

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}.$$

- The scalar product is *associative with respect to product of a scalar and a vector*, that is,

$$(\lambda \boldsymbol{a}) \cdot (\mu \boldsymbol{b}) = (\lambda \mu)(\boldsymbol{a} \cdot \boldsymbol{b})\,.$$

- The scalar product is *distributive*, that is,

  $$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}.$$

  Considering associativity, the following is also true:

  $$\boldsymbol{c} \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{b}.$$

  (This is often referred to as left- and right-distributivity, respectively.)