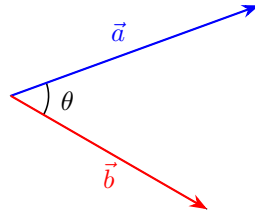


## Scalar product

- The *scalar product* of two vectors  $\vec{a}$  and  $\vec{b}$  yields a *scalar*.
- We denote it by  $\vec{a} \cdot \vec{b}$ .
- It is defined to be the product of the *magnitudes* of  $\vec{a}$  and  $\vec{b}$  and the *cosine* of the smallest angle  $\theta$  between their forward directions, that is,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta. \quad (1)$$



- The scalar product of a vector with itself yields

$$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \cos 0^\circ = |\vec{a}|^2.$$

Therefore, the scalar product of a vector with itself yields the square of the magnitude of the vector. The scalar product of a vector of itself is also denoted as

$$\vec{a}^2 := \vec{a} \cdot \vec{a}. \quad (2)$$

(Note that this is the only time a power is allowed with a vector; for example,  $\vec{a}^3$  or  $\vec{a}^n$  has no meaning.)

- The scalar product of two vectors  $\vec{a}$  and  $\vec{b}$  is equal to zero if and only if

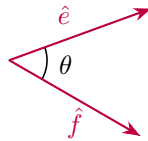
$$\vec{a} = \vec{0} \quad \text{or} \quad \vec{b} = \vec{0} \quad \text{or} \quad \theta = 90^\circ.$$

If  $\theta = 90^\circ$ , then  $\vec{a}$  is perpendicular to  $\vec{b}$  and we denote this fact by  $\vec{a} \perp \vec{b}$ .

- If  $\hat{e}$  and  $\hat{f}$  are two unit vectors and  $\theta$  is the angle between their forward directions, then

$$\hat{e} \cdot \hat{f} = |\hat{e}||\hat{f}| \cos \theta = \cos \theta$$

(since  $|\hat{e}| = 1$  and  $|\hat{f}| = 1$ ).



- It follows for the Cartesian unit vectors that

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

and

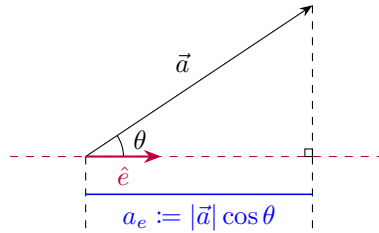
$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

since, for example,

$$\begin{aligned}\hat{x} \cdot \hat{x} &= |\hat{x}| |\hat{x}| \cos 0^\circ = 1, \\ \hat{x} \cdot \hat{y} &= |\hat{x}| |\hat{y}| \cos 90^\circ = 0.\end{aligned}$$

- The component of  $\vec{a}$  in the direction of the vector  $\hat{e}$  can be determined by using the scalar product as follows:

$$\begin{aligned}\vec{a} \cdot \hat{e} &= |\vec{a}| |\hat{e}| \cos \theta \\ &= |\vec{a}| \cos \theta \\ &= a_e.\end{aligned}$$



- The scalar product of two vectors  $\vec{a}$  and  $\vec{b}$  in the Cartesian components of these two vectors is

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z,$$

where

$$\begin{aligned}\vec{a} &= a_x \hat{x} + a_y \hat{y} + a_z \hat{z}, \\ \vec{b} &= b_x \hat{x} + b_y \hat{y} + b_z \hat{z}.\end{aligned}$$

### Algebraic properties

- The scalar product is *not closed* (this means that the product does not yield a vector).
- The scalar product is *commutative*, that is,

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

- The scalar product is *associative with respect to product of a scalar and a vector*, that is,

$$(\lambda \vec{a}) \cdot (\mu \vec{b}) = (\lambda \mu) (\vec{a} \cdot \vec{b}).$$

- The scalar product is *distributive*, that is,

$$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}.$$

Considering associativity, the following is also true:

$$\vec{c} \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b}.$$

(This is often referred to as left- and right-distributivity, respectively.)