Chapter 2

Statics

2.1 Mechanics

Mechanics is the branch of the physical sciences in which the state of *rest* or *motion* of bodies *subject to the action of forces* is investigated. It is the oldest physical science. The earliest recorded writings in this field are those of *Archimedes* (287 - 212 BC), in which the principle of the lever and the principle of buoyancy are described. Substantial progress was made by *Stevinus* (1548 - 1620) who formulated the vectorial combination of forces as well as the principles of statics. *Galileo* (1564 - 1642) was the first person to solve a dynamic problem, namely one involving falling bodies.

The largest and most fundamental contribution to mechanics was made by *Newton* (1642 - 1727) who formulated the laws of motion as well as the law of gravitation and developed the infinitesimal calculus as the mathematical tool used to describe the motion of bodies. This monumental creation is justifiably regarded as the origin of modern science, and many people regard Newton as the greatest scientist of all time.

Many other contributions were made in the period between the fifteenth and nineteenth centuries. In fact, most mathematicians and scientists (and even some artists) dedicated their time to this field, for example *Da Vinci*, *Varignon*, *Euler*, *D'Alembert*, *Lagrange*, *Laplace*, *Hamilton*, *Jacobi* and many others.

Interest in mechanics decreased at the end of the previous century, but there has been a revival in this century, led particularly by the French mathematicians *Poincaré* and *Painlevé*. The UJ participates in this revival, and research of international standard in the *dynamics of non-linear systems* is being done here.

2.1.1 Basic concepts

A number of concepts are of fundamental interest to the study of mechanics, and although they are not all strictly definable, it is necessary to have an intuitive understanding thereof.

- *Space* is the geometric region area occupied by bodies and in which their positions can be established by means of a reference system.
- *Time* is measured by a watch.
- *Mass* is a measure of the inertia of a body, that is, its resistance to a change of velocity.
- *Force* is the mutual influence of bodies on one another.
- A *particle* is a body that can be regarded as small enough in a given situation to be described as a mass at a geometric point.
- A *rigid body* is a body, the deformation of which is negligible.

Mechanics is divided logically into two parts:

- *statics*, which concerns the equilibrium of bodies under the influence of forces; and
- *dynamics*, which concerns the motion of bodies under the influence of forces.

2.1.2 Fields of Application of Mechanics

Although mechanics assumes a fundamental role in our understanding of physical reality and can therefore be studied as a basic science, it takes on further importance if we pay attention to the applicability thereof in engineering. Some of the modern fields of application include the following: satellite movement, fluid dynamics, theory of vibration, stability and strength of structures and machines, robotics, design of spacecrafts, and automatic control and operation of machines.

2.1.3 Limitations of Classical Mechanics

In this course, we shall study classical mechanics, which is based on the "commonsense" concepts of space and time as formulated by Galileo as well as the laws of motion as formulated by Newton. Although this view of mechanics is adequate for the overwhelming majority of engineering applications, it is important for scientists—in particular—to note that classical mechanics is no longer valid in certain extreme *régimes*. If a particle moves at a velocity that is comparable to that of light, for instance, its behaviour is described by *relativistic* mechanics (developed by Einstein in the first two decades of the 20th century), whereas the behaviour of a microscopic particle (that is, the diameter of which is in the order of that of an atom, that is, 10^{-10} metres) is described by *quantum* mechanics (developed by a large number of physicists in the starting in the middle of the 20th century).

2.2 Newton's laws

Classical mechanics as a whole is based on three laws of motion postulated by Isaac Newton in his *Principia Mathematica* of 1686 (the title page is reproduced in Figure 2.1). These laws state the following (we shall in future for instance refer to the first law as **N1**):

Newton 1 A particle remains at rest or continues to move in a straight line with a constant velocity if the resultant force on it is zero.

Newton 2 The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

Newton 3 Two bodies exercise mutual forces on each other, equal in magnitude but opposed in direction.

These three laws have been verified by a large number of accurate experiments. **N2** forms the basis of the analysis in dynamics. For a particle with mass m, which moves under the influence of a force $\boldsymbol{F}$, its acceleration $\boldsymbol{a}$ can be obtained from the algebraic formulation of this law:

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{2.1}$$

It is important to note the *vector nature* of **N2**: not only are the magnitudes of $\boldsymbol{F}$ and $m\boldsymbol{a}$ the same, but $\boldsymbol{F}$ and $\boldsymbol{a}$ also have the same direction. We shall study the dynamics of particles, and therefore the applications of **N2**, in the second semester Applied Mathematics course.

In order to formulate statics, **N1** and **N3** are important. **N1** sets the requirements with which forces must comply to maintain a particle in equilibrium, so it forms the basis of the investigation of static systems. Strictly speaking, **N1** is a special case of **N2**: If $\boldsymbol{F} = \mathbf{0}$ in (2.1), it follows immediately that $\boldsymbol{a} = \mathbf{0}$. **N3** describes the mutual influence of two bodies on each other. This states the fact that if body A exercises a force $\mathbf{F}$ on body B, then body B will exercise a force of $-\boldsymbol{F}$ on body A; that is, a force that has the same magnitude as $\boldsymbol{F}$, but a direction that is opposed to that of $\boldsymbol{F}$. **N3** is therefore very important in the investigation of the equilibrium of *systems* of particles or bodies.

2.3 Units

Four fundamental quantities occur in mechanics: time, distance, mass and force. The units used to measure these quantities cannot be chosen independently because they must be consistent with (2.1). In this course, we shall use the modern metric system of units (the SI system) throughout, which measures distance in metres, mass in kilograms, time in seconds and force in Newton. It is interesting to note the most recent definitions of each of these units:

- A *second* (abbreviation s) is the duration of 9 192 631 770 periods of the radiation associated with the transition between two specific hyper fine levels of the cesium^{133} atom.

- A *metre* (abbreviation m) is the distance that light travels *in vacuo* in 1/299 792 458 of a second.

- A *kilogram* (abbreviation kg) is the mass of a specific cylinder, made of a platinum-iridium alloy, which is kept at the International Bureau of Weights and Measures in a safe in Sévres, France.

- A *Newton* (abbreviation N) is the force required to impart an acceleration of one metre per second per second to a mass of one kilogram.

2.4 Forces

2.4.1 Point of Application

The *experimental* fact that force is a *vector* quantity has already been accepted implicitly in the formulation of the Newton laws. Therefore, we can in future represent a force by means of a directed line segment; the length of the line segment will give us the magnitude of the force in Newton on a chosen scale, and the direction of the line segment will indicate the direction of the force in a given reference system. It is, however, important to note that the mechanical effect of a force does not depend only on its magnitude and direction. Figure 2.2 shows a wheel that can turn freely on an axis through O. If the force $\boldsymbol{F}$ is applied consecutively at points A, O and B respectively, we see immediately that the mechanical effect of the force is different in all three cases; that is, the effect is a clockwise rotation, no rotation and an anti-clockwise rotation respectively. Although the forces are in fact *equal* in these three cases, they are not (mechanically) *equivalent*. The point where a force is applied to a body (like A, B and O above), we call the *point of application* of the force. A straight line through the point of application and parallel to the force is called the *line of action* of the force. In Figure 2.3, the line of action of force $\boldsymbol{F}$ with its point of application at O is indicated by a dotted line. In view of the above, we can now state a very important fact: *A force is only fully specified by the provision of its magnitude and direction as well as the position of its point of application.*

Figure 2.2

Figure 2.3

2.4.2 Classification of Forces

The equilibrium of a body that is subject to a system of forces can be studied only if we know how forces occur. Therefore, it is necessary to undertake a broad classification of the different types of force. We shall describe a number of forces that occur regularly in actual cases; for the sake of simplicity, we shall limit ourselves provisionally to cases in which all forces occur in two dimensions.

1. A *field force* is a force that a body experiences because it resides in the force field of another body. Examples of this are: the gravitational field of any object with mass, the electrostatic field of a charged body and the magnetic field of a moving electrical charge. An important characteristic of these forces is that they cannot be "switched off"; the effect thereof on a body may never be ignored. In this course, we shall pay special attention to the gravitational

field of the earth: any body near the surface of the earth always experiences a force $\boldsymbol{W}$ that is directed at the centre of the earth [Figure 2.4(a)]. We shall in future refer to $\boldsymbol{W}$ as the weight of a body.

2. Forces can be exerted on bodies by means of *ropes* or *rods* [Figure 2.4(b)], in which case the rope or rod is the medium that transfers the force. The point of application of the force obviously is the point where the rope or rod is attached to the object, and the direction of the force is that of the rope or rod.

3. A *spring* that is stretched [as in Figure 2.4(c)] or compressed exercises a restorative force, that is, one that attempts to neutralise the deformation. A spring that is stretched exercises a "pull" and a spring that is compressed exercises a "push". We shall only consider cases in which such a spring is in a linear *régime*, that is, where the *magnitude* of the force in the spring for a deformation x is given by

$$F = kx, \tag{2.2}$$

where the *spring constant* k is a measure of the rigidity of the spring.

4. *Reaction forces* are forces exercised on a body by its contact surface with other objects in its environment. The body in Figure 2.4(d) is on a smooth surface, for instance; to balance its weight in equilibrium, the contact surface must of necessity exercise a *normal* reaction force $\boldsymbol{N}$ (that is, one perpendicular to the contact surface) on the object. Other possibilities are shown in Figure 2.4(e), which shows a truss attached to a wall. Since joint A rests on rollers, the reaction $\boldsymbol{N}$ at A can only be perpendicular to the wall. Joint B is fixed; the reaction force $\boldsymbol{R}$ on it can have any direction and will therefore have general projections $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, which are respectively perpendicular and parallel to the wall.

5. If the object in Figure 2.4(b) were on a rough surface and subjected to the horizontal force of a rope (for instance), we know from everyday experience that there would be a resistance against the horizontal movement of the object. The force responsible for this is called the *friction force*. It has the

following important properties:

(a) The friction force is *tangential* to the contact surface and has a direction that is *opposed* to the direction of the movement or imminent movement of the body. (In other words, a friction force always opposes its cause.)

(b) Only sufficient friction acts to prevent the movement for as long as it is possible.

(c) The friction force $\boldsymbol{f}$ reaches a *maximum* value the moment that the object *is about to start moving.* Experiments have proven that this maximum value is proportional to the magnitude of the *normal reaction force* between the object and the surface. In Figure 2.4(f), the following applies:

$$f_{\max} = \mu_s N, \tag{2.3}$$

where μ_s is called the *static friction coefficient.* If there is movement of the surfaces relative to each other, a *smaller* friction coefficient μ_k, called the *kinetic friction coefficient*, begins to act. Such a coefficient is material dependent, that is, it is a property of the two surfaces that are in contact with each other. Typical values are, for instance, $\mu_s = 0.78$ and $\mu_k = 0.42$ for steel on steel, and $\mu_s = 0.04$ and $\mu_k = 0.04$ for teflon on teflon. Since we are only concerned with the static friction coefficient in statics, we shall indicate the latter by μ only. The following applies under most conditions:

$$0 \le \mu \le 1. \tag{2.4}$$

(d) Experiments have shown that μ is constant and that it does not depend on the size of the contact surface. In Figure 2.4(f) $\boldsymbol{N}$ is the normal reaction force and $\boldsymbol{f}$ is the friction force. Angle ϕ between the *resultant* reaction force $\boldsymbol{R}$ and the normal reaction force is called the *friction angle*, and it follows from the figure that

$$\tan\phi = \frac{f}{N}.$$

But from (2.3) $f = \mu N$, so that

$$\tan \phi = \mu. \tag{2.5}$$

2.5 Axioms

We shall now focus on particles and bodies in equilibrium, that is, in a state of rest or uniform motion. According to **N1**, a *particle* will be in equilibrium if the resultant force on it is zero. If more than one force is exerted on a particle, it is essential to establish how these forces add up to a single force. We do this summation by using two *axioms*, which can be confirmed through experiment:

Axiom 2.1 Two forces are equivalent (that is, they have the same mechanical effect) if and only if they are equal vectors with coinciding lines of action.

This axiom implies that the point of application of a force can be moved along its line of action without affecting its mechanical effect. In figure 2.5, $\boldsymbol{F}$'s mechanical effect is the same whether it is applied at A or at B.

Figure 2.5

Axiom 2.2 If forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied at point P, they can be replaced by the equivalent force $\boldsymbol{F}_1 + \boldsymbol{F}_2$ which is applied at P.

In figure 2.6 $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are exerted on O and O' respectively. According to axiom 2.1, their points of application can be shifted to point of intersection P of their working lines. According to axiom 2.2, they can be added up at P to the single force $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$. We shall refer to $\boldsymbol{F}$ as the *resultant* of the original system of forces. It must again be emphasised that $\boldsymbol{F}$ is only specified fully if the position of its point of application (P in this case) is given.

Figure 2.6

2.6 Equilibrium of a particle

2.6.1 Condition of Equilibrium

In the case of a particle acted on by n forces $\{\boldsymbol{F}_i;\ i = 1, 2, \ldots, n\}$ (Figure 2.7), all the forces and hence their resultant have the same point of application. According to **N1** and Axiom 2.2, the condition of equilibrium of a particle can therefore be formulated as follows:

A particle is in equilibrium if only and only if the vector sum of the forces acting on the particle is the null vector.

Hence for the forces that maintain the particle in Figure 2.7 in equilibrium, the following applies:

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots + \boldsymbol{F}_n = \boldsymbol{0}. \tag{2.6}$$

Figure 2.7

Since all the components of the null vector are equal to zero and since the ξ component, for instance, of a sum of vectors is equal to the sum of the vectors'

ξ components (see §1.7.3), (2.6) is equivalent to a system of three simultaneous equations:

$$\begin{aligned} \sum F_x &= F_{x1} + F_{x2} + \cdots + F_{xn} = 0 \\ \sum F_y &= F_{y1} + F_{y2} + \cdots + F_{yn} = 0 \\ \sum F_z &= F_{z1} + F_{z2} + \cdots + F_{zn} = 0 \end{aligned} \tag{2.7}$$

In (2.7) F_{xk}, F_{yk} and F_{zk} are the x, y and z components of $\boldsymbol{F}_k$ respectively.

In cases where a particle is in interaction with other objects, we must in a given problem decide at the outset which forces are exerted *on* the particle by its environment. In such cases, we often say we "isolate" the particle, that is, its environment is replaced by the forces that the environment exerts on it (**N3** often is useful in this regard).

Examples

V2.6.1. A particle is in equilibrium and under the influence of five coplanar forces. The magnitudes of the first four are 2, 4, $6\sqrt{3}$ and 8 Newton, respectively, and the directions of the last three form angles of 60°, 150° and 300° respectively with those of the first force. Calculate the fifth force.

Solution: Unknown force $\boldsymbol{F}_5$ (Figure 2.8) is obtained from **N1**:

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4 + \boldsymbol{F}_5 = \mathbf{0}$$

that is,

$$\boldsymbol{F}_5 = -(\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4)$$

A reference system, the X axis of which coincides with $\boldsymbol{F}_1$ is introduced; this

Figure 2.8

choice enables us to write out the known forces in component form:

$$\begin{aligned}\boldsymbol{F}_5 &= [-2\hat{x} - 4(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}) - 6\sqrt{3}(-\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}) - 8(\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y})] \text{ N} \\ &= (\hat{x} - \sqrt{3}\hat{y}) \text{ N}\end{aligned}$$

$\boldsymbol{F}_5$ can now be represented graphically (Figure 2.9) and its magnitude and direction follow from elementary vector algebra:

$$F_5 = \sqrt{1^2 + \left(\sqrt{3}\right)^2} \text{ N} = 2 \text{ N}$$

and

$$\tan\alpha = \frac{\sqrt{3}}{1} \quad \Rightarrow \quad \alpha = 60^\circ.$$

V2.6.2. A particle (weight W) is suspended in the earth's gravitational field, as shown in Figure 2.10. Calculate the forces in the ropes.

Solution: First we identify the forces that are exerted on the particle. We know both the magnitude and direction of weight $\mathbf{W}$ of the particle as well as the directions of the forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ in the ropes. We only need to find their

Figure 2.9

Figure 2.10

magnitudes. We can determine the magnitudes of the forces easily by graphic means by exhibiting the condition of equilibrium

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{W} = \mathbf{0}$$

in the vector diagram in Figure 2.11. It follows directly from the diagram that

$$\begin{aligned} T_1 &= W \cos 30^\circ = \frac{\sqrt{3}}{2} W \\ \text{and} \quad T_2 &= W \cos 60^\circ = \frac{1}{2} W. \end{aligned}$$

V2.6.3. In Figure 2.12, the indicated mass (weight W) rests on a rough inclined surface with friction coefficient μ, and it is just about to move *upwards* under the

Figure 2.11

influence of the external force $\boldsymbol{P}$, the direction of which is shown. Calculate the magnitude of $\boldsymbol{P}$.

Figure 2.12

Solution: We can identify four forces acting on the particle: its weight $\boldsymbol{W}$ and the applied force $\boldsymbol{P}$ (we know both their magnitude and direction), the normal reaction force $\boldsymbol{N}$ (of which we know only the direction) and the friction force $\boldsymbol{f}$ (directed *downwards* along the surface because it *opposes* $\boldsymbol{P}$ and the magnitude of which is known up to the unknown value of $\boldsymbol{N}$'s magnitude: $f = \mu N$). The condition of equilibrium **N1** is then as follows:

$$\boldsymbol{P} + \boldsymbol{W} + \boldsymbol{f} + \boldsymbol{N} = \mathbf{0}.$$

Since all the components of the null vector are zero, the x and y components, among others, of the above vector sum will be zero for the choice of reference system in Figure 2.12. We can calculate these two components by bearing in mind that the x component of the vector sum is given, for instance, by the sum of the x components of the vectors in the sum:

$$x-\text{component:} \qquad P\cos\beta - W\sin\alpha - \mu N = 0 \qquad \text{(a)}$$
$$y-\text{component:} \qquad P\sin\beta + N - W\cos\alpha = 0 \qquad \text{(b)}$$

It follows from (b) that

$$N = W\cos\alpha - P\sin\beta.$$

We substitute this expression for N in (a):

$$P\cos\beta - W\sin\alpha - \mu W\cos\alpha + \mu P\sin\beta = 0.$$

We can then make the magnitude of the external force the subject of the equation to find

$$P = W\left(\frac{\sin\alpha + \mu\cos\alpha}{\cos\beta + \mu\sin\beta}\right).$$

Problems

P2.6.1. The following forces are exerted on a particle:

(a) 10 N, parallel to $\hat{x}+\hat{y}$;

(b) 15 N, with direction cosines 0.0, -0.8 and 0.6;

(c) $(3\hat{x}+15\hat{y}+\hat{z})$ N;

(d) 5 N parallel to the negative X axis;

(e) 5 N forming angles of 90°, 135° and 135° with the positive directions of the coordinate axes respectively.

Calculate the magnitudes and direction cosines of the resultant of the above forces.

P 2.6.2. Figure 2.13 shows a particle with weight W hung from two ropes, the lengths of which are given. Calculate the forces in the ropes.

Figure 2.13

P2.6.3. Figure 2.14 shows a particle with weight W suspended from three rods, the lengths of which are given. The complete geometry of the setup is shown with relationship to the chosen reference system. Calculate the forces in all three rods.

Figure 2.14

P2.6.4. In Figure 2.15 a mass of weight W rests on a smooth surface and is kept in equilibrium by a spring with force constant k. Calculate the extension of the spring.

P2.6.5. Calculate the magnitude of $\boldsymbol{P}$ in V2.6.3 if the mass is just about to slide *downwards*.

P2.6.6. A mass with weight W rests on a rough horizontal surface and is pulled by a force $\boldsymbol{F}$, which forms angle β with the horizontal. If the friction coefficient

Figure 2.15

is μ, establish for which value of β the magnitude of the force will assume a *minimum* value when the body is just about to move. (Hint: Approach the problem graphically.)

P2.6.7. Answer P2.6.6 for the case in which the surface on which the mass rests forms angle α with the horizontal and the mass is about to move:

(a) upwards and

(b) downwards along the surface.

2.6.2 Coupled Particles

Particles are often coupled to each other by means of ropes, rods or springs, in which case the condition of equilibrium in §2.6.1 obviously has to be applied *separately* to *each* of the particles in question. A further complication which then occurs is that the relevant particles exert forces on each other with a rope or spring as the medium, and these forces are often unknown *a priori*. The way in which this complication is handled is by isolating *each* particle and by bearing in mind **N3** when handling the rope forces, for instance. We illustrate this process on the basis of the simple situation in Figure 2.16: Two particles linked by means of a rope are in equilibrium under the influence of force $\boldsymbol{F}$. **N1** requires that A exerts a force $-\boldsymbol{F}$ on B by means of the rope. According to **N3**, it follows that B exerts a force $\boldsymbol{F}$ on A by means of the rope, etc. We can also immediately draw a conclusion about the force in the rope: this force must be the same everywhere in

Figure 2.16

the rope (every part of the rope must also be in equilibrium). We must also note an important difference between ropes, rods and springs here: whereas rods and springs can exert both a “push” and a “pull”, ropes can only exert a “pull”.

Examples

V2.6.4. The rope in figure 2.17 has no mass and the pulley is smooth. The mass B (weight W) is on a horizontal surface on which the friction coefficient is $\frac{1}{3}$ and is about to slide. What is the weight of A?

Figure 2.17

Solution: First, we isolate each of the two particles in the problem, that is, in a sketch (Figure 2.18) we indicate which forces are being exerted *on* each of the particles. Although we know that the rope exerts a pull on both particles and

Figure 2.18

we know the direction of the rope force everywhere, we do not initially know the magnitudes of the forces that the rope exerts on the two particles. The magnitudes of the forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are therefore the unknown elements in this problem. We then write down the conditions of equilibrium of each of the two particles as well as the x and y components (in the shown reference system) of each of the vector equations that are obtained:

$$\begin{aligned} &\text{Equilibrium of B:} && \boldsymbol{T}_1 + \boldsymbol{W} + \boldsymbol{f} + \boldsymbol{N} = \boldsymbol{0} \\ &x - \text{component:} && T_1 - \frac{1}{3}N = 0 && \text{(a)} \\ &y - \text{component:} && N - W = 0 && \text{(b)} \end{aligned}$$

$$\begin{aligned} &\text{Equilibrium of A:} && \boldsymbol{T}_2 + \boldsymbol{P} = \boldsymbol{0} \\ &y - \text{component:} && T_2 - P = 0 && \text{(c)} \end{aligned}$$

The three equations (a), (b) and (c) contain the four unknowns N, T_1, T_2 and P, and more information is required before the problem can be solved fully. This information is contained in the fact that the tensile force in the rope is the same everywhere, that is, that the magnitudes of $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are the same:

$$T_1 = T_2. \qquad \text{(d)}$$

Equations (a) to (d) can now be solved easily for all four unknown quantities:

$$N = W, \quad T_1 = T_2 = \frac{1}{3}W, \quad P = \frac{1}{3}W.$$

Problems

In the following problems, the ropes have no mass and the pulleys are smooth.

P2.6.8. In Figure 2.19, the weight of blocks A and B are W and $3W$ respectively. The rope that links the two blocks goes over a pulley at C. If the friction coefficient between all the surfaces is $\frac{1}{2}$, calculate the magnitude of force $\boldsymbol{F}$ necessary to make block B just move to the right.

Figure 2.19

P2.6.9. The weight of masses A and B in Figure 2.20 is W_1 and W_2 respectively. The friction coefficient μ on both inclined surfaces is the same. Calculate μ if the system is just about to start moving.

P2.6.10. Two identical masses rest on the sides of a double inclined surface, which forms inclined angles 30° and 60°. They are linked by a rope that goes over a pulley at the common apex of the two surfaces. The friction coefficient on both surfaces is the same. Show that the masses are just about to slide if $\mu = 2 - \sqrt{3}$.

P2.6.11. The weight of masses A and B in Figure 2.21 are W_1 and W_2 respectively, and the stiffness constants of the two springs are k_1 and k_2 respectively. Calculate the extension of both springs in equilibrium.

Figure 2.20

Figure 2.21

2.7 Two-dimensional systems of forces

If the physical measurements of a rigid body in a given situation must be taken into account, the forces being exerted on the body can generally have different points of application, and the condition of equilibrium of §2.6.1 is then obviously no longer sufficient for the analysis of the problem. We shall limit ourselves for the time being to the case in which all the forces being exerted on the body are vectors lying on the same plane (that is, a *two-dimensional* space). We will see that we develop concepts for this special case which will enable us to solve the general problem later.

2.7.1 Resultant of Non-parallel Forces

If two non-parallel, coplanar forces are exerted on a body, as in Figure 2.22, their lines of action must of necessity intersect at one point. According to axioms 2.1

Figure 2.22

and 2.2, the two forces at this point can add up to a single force that is *equivalent* to the original system of two forces. In this way, a system of coplanar forces can in general be reduced to a single force by means of repeated pairwise addition.

2.7.2 Resultant of Parallel Forces

The lines of action of two parallel forces do not intersect, in which case axioms 2.1 and 2.2 can not be used. We can, however, with the help of a construction find a solution to this apparent dilemma.

We consider two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, where

$$\boldsymbol{F}_1 = \lambda \boldsymbol{F}_2 \tag{2.8}$$

and where λ is any scalar. In Figure 2.23 the points of application A and B of these two forces are first displaced so that they both lie on a line that is perpendicular to the directions of the forces. Then we add to the system a force $\boldsymbol{P}$, the direction of which is parallel to line BA and which acts at A as well as force $-\boldsymbol{P}$ which acts at B. Since these two forces can be added to the *null force* at any point on AB with the help of axioms 2.1 and 2.2 their addition does not change the mechanical effect of the original system.

Figure 2.23

Forces $\boldsymbol{F}_1$ and $\boldsymbol{P}$ both act on A and can be added there to yield $\boldsymbol{F}_1 + \boldsymbol{P}$; similarly $\boldsymbol{F}_2$ and $\boldsymbol{P}$ can be added at B to yield $\boldsymbol{F}_2 + \boldsymbol{P}$. These two forces obviously are not parallel, and with the help of axioms 2.1 and 2.2 they can be summed to a single force at point O where their lines of action intersect:

$$\boldsymbol{F} = (\boldsymbol{F}_1 + \boldsymbol{P}) + (\boldsymbol{F}_2 - \boldsymbol{P}) = \boldsymbol{F}_1 + \boldsymbol{F}_2. \tag{2.9}$$

Equation (2.9) determines the *magnitude* and *direction* of the resultant $\boldsymbol{F}$. The line of action of the resultant follows from the geometry of the construction. Since $\triangle ACD$ and $\triangle BEG$ are similar to $\triangle OHA$ and $\triangle OHB$ respectively, it follows that

$$\frac{F_1}{P} = \frac{q}{x_1} \quad \text{and} \quad \frac{F_2}{P} = \frac{q}{x_2}$$

so that

$$Pq = F_1 x_1 \quad \text{and} \quad Pq = F_2 x_2$$

and hence

$$\frac{x_1}{x_2} = \frac{F_2}{F_1}. \tag{2.10}$$

Since (2.10) does not contain P or q, this result is independent of the details of the construction, and the line of action of $\boldsymbol{F}$ is uniquely determined.

Problems

P 2.7.1. Redraw Figure 2.23 for the case in which the directions of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are opposed, and calculate a result similar to (2.10) for the line of action of the resultant.

2.7.3 The Couple

The question arises immediately whether the construction in §2.7.2 produces a result in all instances. Obviously the construction cannot succeed if the lines of action of forces $\boldsymbol{F}_1 + \mathbf{P}$ and $\boldsymbol{F}_2 - \boldsymbol{P}$ in Figure 2.23 are also parallel, that is, if

$$\boldsymbol{F}_1 + \boldsymbol{P} = \alpha(\boldsymbol{F}_2 - \boldsymbol{P}).$$

From (2.8) we then have

$$\lambda \boldsymbol{F}_2 + \boldsymbol{P} = \alpha(\boldsymbol{F}_2 - \boldsymbol{P})$$

and hence

$$(\lambda - \alpha)\boldsymbol{F}_2 = -(\alpha + 1)\boldsymbol{P}.$$

Since $\boldsymbol{F}_2$ and $\boldsymbol{P}$ are not parallel, the last equation can be true only if the null vector occurs on both sides of the equation, that is, if

$$\lambda - \alpha = 0 \quad \text{and} \quad \alpha + 1 = 0$$

so that

$$\lambda = \alpha = -1.$$

For $\lambda = -1$, we have a system of *two opposed forces with different lines of action* in Figure 2.23. We shall refer to such a system (Figure 2.24) as a *couple.*

A couple can never be reduced to a single force.

2.7.4 Reduction of Two-dimensional System of Forces

It follows immediately from the above that a two-dimensional system of forces can be reduced *either to a force or to a couple.*

Figure 2.24

2.8 Moments

2.8.1 Definition

Experiments have shown that a couple has a *purely rotational effect* on a body. We know from experience that, in general, any force has two effects on a body: firstly, it tends to make the body *translate* (or move uniformly) in the direction of the force and secondly, it tends to make the body *rotate* about an axis. To be able to analyse the equilibrium problem of a rigid body further, it is necessary to express the rotational effect of forces quantitatively.

Figure 2.25 is a schematic representation of a spanner OS that is used to turn a nut on a bolt at O. Force $\boldsymbol{F}$ is in the same plane as the spanner. From everyday

Figure 2.25

experience and from experiments we know that the rotational effect on O of force

$\boldsymbol{F}$ being exerted on the spanner can be increased in two ways: firstly $\boldsymbol{F}$ can be increased and secondly, d, that is, the perpendicular distance between O and $\boldsymbol{F}$'s line of action, can be increased. It is natural to use the product of these two quantities as a measure of the rotational effect of $\boldsymbol{F}$ about O:

$$M_O = Fd \tag{2.11}$$

The moment of $\boldsymbol{F}$ on O, M_O, is defined in (2.11). However, the rotational effect has a *preferential direction*, which is about an axis *perpendicular* to the plane that contains $\boldsymbol{F}$ and $\boldsymbol{r}$ (the position vector of $\boldsymbol{F}$'s point of applications with regard to O). Therefore, it is natural to regard the moment as a *vector*, the magnitude of which is given by (2.11) and the direction of which is that of the preferential rotational axis. Since we can describe (2.11) from Figure 2.25 as

$$M_O = Fr\sin\theta = |\boldsymbol{r}\times\boldsymbol{F}|\,,$$

the complete definition of the *moment* of $\boldsymbol{F}$ about O is:

$$\boldsymbol{M}_O = \boldsymbol{r}\times\boldsymbol{F} \tag{2.12}$$

2.8.2 Dimensions

It follows immediately from this definition that the dimensions of a moment are the product of the dimensions of force and distance. In the SI system, we have $[M] = \mathrm{Nm}$.

2.8.3 Moment of Equivalent Forces

If the point of application of $\boldsymbol{F}$ in Figure 2.25 is displaced to R, we know from axiom 2.1 that its mechanical effect remains the same. However, it follows from (1.38) that $\boldsymbol{M}_O = \boldsymbol{r}\times\boldsymbol{F} = \boldsymbol{d}\times\boldsymbol{F}$, and the moment of $\boldsymbol{F}$ about O will remain the same if its point of application is shifted along its line of action. Therefore, we see that *equivalent forces have the same moment about a given point.*

2.8.4 The Vector Nature of Moment

Although the quantity that we defined in (2.12) has a magnitude and a direction, we can regard it as a vector quantity only if it complies with the algebraic rules of §1.5. Experiments have shown that this is indeed true.

Of particular importance is the fact that moments sum like vectors, in other words: *the resultant rotational effect of a number of forces about a given point is given by the sum of their moments about this point.* In Figure 2.26, the total moment about O of the depicted system of forces is:

$$\begin{aligned} \boldsymbol{M}_O &= \boldsymbol{M}_{O,1} + \boldsymbol{M}_{O,2} + \cdots + \boldsymbol{M}_{O,n} \\ &= \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 + \cdots + \boldsymbol{r}_n \times \boldsymbol{F}_n \\ &= \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i. \end{aligned} \tag{2.13}$$

Figure 2.26

Examples

V2.8.1. In Figure 2.27, a force $\boldsymbol{P}$ is applied on a lever. Calculate the moment of $\boldsymbol{P}$ about O.

Solution: Since we have a two-dimensional problem here, we can calculate M_O directly from (2.11); the direction of the moment is perpendicular to the plane that contains $\boldsymbol{P}$ and $\boldsymbol{r}$, and can be obtained from (2.12) with the help of the right-hand

Figure 2.27

rule.

$$M_O = Pd = Pa \sin 20° = 0.342Pa$$

Therefore, for the chosen reference system in Figure 2.27 we have:

$$\boldsymbol{M}_O = 0.342Pa(-\hat{z}).$$

V2.8.2. The length of mast AB in Figure 2.28 is 6 m, and it is kept in equilibrium by three cables, as shown. The magnitude of force $\boldsymbol{T}$ in cable BE is 840 N.

Figure 2.28

Calculate the moment of this force about C.

Solution: Since the plane in which $\boldsymbol{s}$ (the position vector of $\boldsymbol{T}$'s point of application with regard to C) and $\boldsymbol{T}$ lie is not one of the coordinate planes, we must use definition (2.12):

$$\boldsymbol{M}_C = \boldsymbol{s} \times \boldsymbol{T}.$$

The vector product can be calculated easily if we can express $\boldsymbol{s}$ and $\boldsymbol{T}$ in component form. It follows immediately from the geometry of Figure 2.28 that:

$$\begin{aligned} \boldsymbol{s} &= (3-0)\hat{x} + (6-2)\hat{y} + (0-3)\hat{z} \text{ m} \\ &= 3\hat{x} + 4\hat{y} - 3\hat{z} \text{ m}. \end{aligned}$$

We know the magnitude of $\boldsymbol{T}$ ($T = 840$ N) and its direction is the same as that of *displacement* BE. The latter can also be described in component form from the geometry of the setup:

$$\begin{aligned} \overline{BE} &= (6-3)\hat{x} + (0-6)\hat{y} + (2-0)\hat{z} \text{ m} \\ &= 3\hat{x} - 6\hat{y} + 2\hat{z} \text{ m}. \end{aligned}$$

We then construct a unit vector $\hat{n}$ with the same direction as $\overline{BE}$,

$$\hat{n} = \frac{\overline{BE}}{|\overline{BE}|} = \frac{1}{7}(3\hat{x} - 6\hat{y} + 2\hat{z}),$$

after which we can write $\boldsymbol{T}$ in component form:

$$\begin{aligned} \boldsymbol{T} = T\hat{n} &= \frac{840}{7}(3\hat{x} - 6\hat{y} + 2\hat{z}) \text{ N} \\ &= 360\hat{x} - 720\hat{y} + 240\hat{z} \text{ N} \end{aligned}$$

It then follows for the required moment that

$$\begin{aligned} \boldsymbol{M}_C &= \boldsymbol{s} \times \boldsymbol{T} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 4 & -3 \\ 360 & -720 & 240 \end{vmatrix} \text{ Nm} \\ &= -1200\hat{x} - 1800\hat{y} - 3600\hat{z} \text{ Nm}. \end{aligned}$$

Therefore, the magnitude of $\boldsymbol{M}_C$ is $M_C = 2\ 200$ Nm and its direction is that of unit vector $\hat{M}_C = \frac{1}{7}(-2\hat{x} - 3\hat{y} - 6\hat{z})$.

Problems

P2.8.1. A force of 400 N is applied to end B of a pipe in Figure 2.29. Calculate

(a) the moment of this force about A, and

(b) the magnitude and direction of a horizontal force, exerted on C, which has the same moment about A.

Figure 2.29

P2.8.2. A force $\boldsymbol{F}$ of 80 N is applied to A on the handle of the guillotine in Figure 2.30.

(a) Calculate the moment of this force about hinge O for $\theta = 60°$.

(b) For what value of θ will $\boldsymbol{F}$'s (clockwise) moment on O assume a maximum value?

P2.8.3. In Figure 2.31, crank A is prevented by the key at B to slide on fixed axis s. Calculate the moment of the 800 N force about B.

P2.8.4. The mast in Figure 2.32 supports a traffic light with weight W. Calculate the moment on the weight about A.

Figure 2.30

Figure 2.31

P2.8.5. A force $\boldsymbol{F} = P(\hat{x} + 2\hat{y} - \hat{z})$ is exerted on D in Figure 2.33. Calculate the moments of this force about the joints at B and C.

P 2.8.6. Mast AB in Figure 2.34 lies in the YZ plane and a tensile force of magnitude $F = 500$ N is applied to it by cable BC.

(a) Calculate the moment of the tensile force about A.

(b) What is the shortest distance from A to the cable?

Figure 2.32

Figure 2.33

2.9 Couples

2.9.1 Moment of a Couple

We defined a couple in §2.7.3 as two opposite forces with different working lines; we shall henceforth for instance indicate the couple in Figure 2.35 by $\{\boldsymbol{F}, -\boldsymbol{F}\}$. If we consider an *arbitrary point* P, both the forces in a given couple will have moments on P. From (2.13), we can obtain the total moment $\boldsymbol{M}_P$ of the couple about P by summing these moments, and in Figure 2.35 it then follows that

$$\begin{aligned}\boldsymbol{M}_P &= \boldsymbol{s}_A \times \boldsymbol{F} + \boldsymbol{s}_B \times (-\boldsymbol{F}) \\ &= (\boldsymbol{s}_A - \boldsymbol{s}_B) \times \boldsymbol{F} \\ &= \boldsymbol{s} \times \boldsymbol{F}.\end{aligned}$$

Figure 2.34

Since $\boldsymbol{s}$ is the displacement between the points of application of the forces $\boldsymbol{F}$ and

Figure 2.35

$-\boldsymbol{F}$, $\boldsymbol{M}_P$ is independent of the position of P, in other words: *the moment of a couple about all points is the same*. Therefore, we can now omit the subscript P in the above and we need to refer only to "the moment of the couple". We can calculate the magnitude and direction of this moment from the geometry in Figure 2.35 by making use of (1.48):

$$\boldsymbol{M} = \boldsymbol{s} \times \boldsymbol{F} = \boldsymbol{d} \times \boldsymbol{F} = (Fd)\hat{n} \tag{2.14}$$

In (2.14), $\hat{n}$ is perpendicular to the plane that contains $\{\boldsymbol{F}, -\boldsymbol{F}\}$ and its direction is found by means of the right-hand rule from the couple's sense of rotation. Therefore, we see that the *magnitude* of a couple's moment is the product of the *magnitude of the forces in the couple and the distance between the forces*, whereas the *direction* of the moment is *perpendicular to the plane that contains the two forces* and can be obtained by means of the right-hand rule from the couple's sense of rotation.

2.9.2 Equivalence of Couples

One can obviously construct more than one couple with a given moment. The question that arises immediately is whether or not $\boldsymbol{M}$ in §2.9.1 is a *free* vector, that is, whether or not all the couples with the same moment are *mechanically equivalent*. We can answer this question by realising firstly that *three* operations can be carried out on a couple *without changing its moment*:

(a) Increase the magnitude of the forces in the couple by a factor α and the distance between them by the factor $\frac{1}{\alpha}$.

(b) Rotate the couple in the plane which contains it.

(c) Translate the entire couple without changing the magnitudes or directions of the forces.

We now show that in each of these three cases an equivalent couple is obtained.

(i) Couple $\{\boldsymbol{F}, -\boldsymbol{F}\}$ with points of application at A and B is shown in Figure 2.36; with the help of axiom 2.1, these points can be chosen such that AB is perpendicular to both forces. The *null force* is now added to the system by letting

Figure 2.36

oppessed forces $-(\alpha-1)\boldsymbol{F}$ and $(\alpha-1)\boldsymbol{F}$ act on Q, the midpoint of AB. From §2.7.2, we can then sum $\boldsymbol{F}$ and $(\alpha-1)\boldsymbol{F}$ to a single force $-\alpha\boldsymbol{F}$ at B'. It follows from (2.10) for the position of A' that

$$F\left(\frac{d}{2}-a\right)=(\alpha-1)Fa$$

and hence

$$a=\frac{d}{2\alpha}.$$

Similarly, it follows for the position of B' that

$$b=\frac{d}{2\alpha},$$

so that the distance between A' and B' is given by

$$d'=a+b=\frac{d}{\alpha}.$$

Therefore, we see that operation (a) above merely amounts to the addition of a *null force* to the original couple, and couple $\{\alpha\boldsymbol{F},-\alpha\boldsymbol{F}\}$, which is obtained by means of this operation, is equivalent to the original one.

(ii) Figure 2.37 shows couple $\{\boldsymbol{F},-\boldsymbol{F}\}$; the points of application A and B are again chosen so that AB is perpendicular to the forces. Line segment AB is now

Figure 2.37

rotated through an arbitrary angle θ about its centre point to obtain line segment

$A'B'$. The null force is then added to A' and B' by allowing both $\boldsymbol{F'}$ and its opposed force $-\boldsymbol{F}'$ to be applied to each of these points, where $\boldsymbol{F'}$ is perpendicular to $A'B'$ and $|\boldsymbol{F'}| = |\boldsymbol{F}|$. Since only the null force is added to the system, the new system of six forces is equivalent to the original couple. With the help of axioms 2.1 and 2.2, forces $\boldsymbol{F}$ at A and $-\boldsymbol{F'}$ at A' can be summed to a single force $\boldsymbol{F} - \boldsymbol{F'}$ which acts at C. Similarly, forces $-\boldsymbol{F}$ at B and $\boldsymbol{F'}$ at B' can be summed to a single force $\boldsymbol{F'} - \boldsymbol{F}$ which acts at D. It now follows from the construction that $\triangle OBD$ and $\triangle OB'D$ is congruent, so that

$$\beta = \gamma = 90^\circ - \frac{\theta}{2}.$$

It then follows for the complete angle about D that

$$2\alpha + 2\theta + 2(90^\circ - \frac{\theta}{2}) = 360^\circ,$$

so that

$$\alpha = 90^\circ - \frac{\theta}{2} = \beta = \gamma.$$

Therefore, we see that α and β are opposite angles, which means that force $\boldsymbol{F'} - \boldsymbol{F}$ has the same direction as OD. We can establish through a similar argument that force $\boldsymbol{F} - \boldsymbol{F'}$ has the same direction as OC. Since DOC is a straight line, $\boldsymbol{F} - \boldsymbol{F'}$ and $\boldsymbol{F'} - \boldsymbol{F}$ have the *same line of action*, and they can be summed on this line of action to the single force $(\boldsymbol{F} - \boldsymbol{F'}) + (\boldsymbol{F'} - \boldsymbol{F}) = \boldsymbol{0}$. The original couple is therefore equivalent to the remaining couple $\{\boldsymbol{F'}, -\boldsymbol{F'}\}$ and we have now shown that operation (b) also produces an equivalent couple.

(iii) Consider couple $\{\boldsymbol{F}, -\boldsymbol{F}\}$ in Figure 2.38(a) with points of application A and B. The null force is now added to an arbitrary point O by applying forces $2\boldsymbol{F}$ and $-2\boldsymbol{F}$ to it. With the help of P2.7.1, forces $\boldsymbol{F}$ at A and $-2\boldsymbol{F}$ at O can be summed to a single force $-\boldsymbol{F}$ at D. The position of D is established on the basis of Figure 2.38(b): According to P2.7.1, we have

$$F(a + b) = (2F)b$$

(a)

(b)

Figure 2.38

and hence

$$a = b.$$

It follows immediately from the similarity of the triangles in Figure 2.38(b) that $AO = OD$. Similarly, it can also be shown that $BO = OC$. Since its diagonals bisect each other, $ABDC$ is a parallelogram, so that $\overline{BA} = \overline{DC}$. By adding the null force, the original couple is therefore translated through space, and operation (c) also produces an equivalent couple.

We have now shown that *any two couples with the same moment are equivalent.* Therefore, the moment of a couple in fact contains complete information about the couple's mechanical effect.

2.9.3 Summation of Couples

Only one question about couples still has to be answered, that is, the joint effect of two or more couples being applied to a body. Consider any two couples with

Figure 2.39

with moments $\boldsymbol{M}_1$ en $\boldsymbol{M}_2$ respectively, where these two moments are not parallel. (In the case of parallel moments, this argument becomes trivial). The two planes in which the forces composing the two couples respectively lie will intersect each other in a straight line. Let $\boldsymbol{s}$ be any directed line segment on this line. With the help of operations (a) to (c) in §2.9.2, the two couples can now be translated so that $\boldsymbol{s}$ is the common separating vector of the points of application of the respective pairs of forces in the two couples (Figure 2.39). Forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ obviously are such that $\boldsymbol{M}_1 = \boldsymbol{s} \times \boldsymbol{F}_1$ and $\boldsymbol{M}_2 = \boldsymbol{s} \times \boldsymbol{F}_2$. Since $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ now have the same point of application B, they can be replaced by an equivalent force $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$ on B. Similarly, $-\boldsymbol{F}_1$ and $-\boldsymbol{F}_2$ at A can be replaced by an equivalent force $-\boldsymbol{F}$. Therefore, we have a resultant couple $\{\boldsymbol{F}, -\boldsymbol{F}\}$ with moment

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{s} \times \boldsymbol{F} \\ &= \boldsymbol{s} \times (\boldsymbol{F}_1 + \boldsymbol{F}_2) \\ &= \boldsymbol{s} \times \boldsymbol{F}_1 + \boldsymbol{s} \times \boldsymbol{F}_2 \\ &= \boldsymbol{M}_1 + \boldsymbol{M}_2. \end{aligned} \tag{2.15}$$

In the third line of (2.15), we used the well-known property (1.49) of the vector product.

We can summarise this important result as follows: *a number of couples are jointly equivalent to a single couple, the moment of which is the vector sum of that of the respective couples.*

2.10 Resultant of System of Forces

In §2.5 we referred to the vector sum (on a specific line of action) of a two-dimensional system of forces as the system's resultant. We will now expand this concept for any system of forces: *the resultant of a system of forces is the most simple system that is mechanically equivalent to it.* With the help of the results in §2.9, we can formulate a theorem that describes this most simple system in general.

2.10.1 Reduction Theorem

Figure 2.40 depicts a system of forces $\{\boldsymbol{F}_i;\ i = 1, 2, \ldots, n\}$ with points of application A_i. Let P be an *arbitrary* point of reference and let $\boldsymbol{s}_i$ be the position vector

Figure 2.40

of A_i $(i = 1, 2, \ldots, n)$ with respect to P. For the i-th force, the *null force* is added to P by letting both $\boldsymbol{F}_i$ en $-\boldsymbol{F}_i$ act on it. In this way, $\boldsymbol{F}_i$ is replaced at A_i by an *equivalent system* that consists of force $\boldsymbol{F}_i$ at P and a couple with moment $\boldsymbol{s}_i \times \boldsymbol{F}_i$. This process is repeated for all the n forces. The n forces at P then add up to *a* *single force* at P:

$$\boldsymbol{F} = \sum_{i=1}^{n} \boldsymbol{F}_i. \tag{2.16a}$$

The n couples add up to *a single couple* with moment

$$\boldsymbol{M}_P = \sum_{i=1}^{n} \boldsymbol{s}_i \times \boldsymbol{F}_i. \tag{2.16b}$$

This theorem is very important for our analysis of systems of forces, so we formulate it in words as follows: the resultant of a system of forces consists in general of a *single force* at an *arbitrary* (that is, freely chosen) point P and a *couple*, where

(a) the force is the *vector sum* of the *forces* in the given system, and

(b) the moment of the couple is the *total moment of the given system about* P.

It is also important to note that since the null force is added to the system to obtain its resultant, the *moment of the resultant about any point is the same as that of the original system.*

Examples

V2.10.1. Replace the system in Figure 2.41 with a force at O and a couple.

Figure 2.41

Solution: This system consists of the four forces $\boldsymbol{F}_1 = 4P(-\hat{z})$, $\boldsymbol{F}_2 = 2P(-\hat{z})$, $\boldsymbol{F}_3 = 4P(-\hat{z})$ and $\boldsymbol{F}_4 = 5P(-\hat{z})$, the position vectors with respect to O of which are given by $\boldsymbol{s}_1 = \boldsymbol{0}$, $\boldsymbol{s}_2 = 3a\hat{x}$, $\boldsymbol{s}_3 = 3a\hat{x} + 2a\hat{y}$ and $\boldsymbol{s}_4 = 2a\hat{y}$ respectively. Therefore, the force at O is

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4 = -15P\hat{z},$$

whereas the moment of the couple is given by:

$$\begin{aligned}\boldsymbol{M}_O &= \boldsymbol{s}_1 \times \boldsymbol{F}_1 + \boldsymbol{s}_2 \times \boldsymbol{F}_2 + \boldsymbol{s}_3 \times \boldsymbol{F}_3 + \boldsymbol{s}_4 \times \boldsymbol{F}_4 \\ &= \boldsymbol{0} \times (-4P\hat{z}) + (3a\hat{x}) \times (-2P\hat{z}) + (3a\hat{x} + 2a\hat{y}) \times (-4P\hat{z}) \\ &\quad + (2a\hat{y}) \times (-5P\hat{z}) \\ &= \boldsymbol{0} + 6Pa\hat{y} + 4Pa(3\hat{y} - 2\hat{x}) - 10Pa\hat{x} \\ &= Pa(-18\hat{x} + 18\hat{y}).\end{aligned}$$

Therefore, $\boldsymbol{M}_O$ is a vector with magnitude $18\sqrt{2}Pa$, which lies in the XY plane and forms an angle of $135°$ with the positive X axis.

2.10.2 Special Case

Theorem 2. If

$$\boldsymbol{F} \cdot \boldsymbol{M}_P = 0 \tag{2.17}$$

in (2.16) the system can be replaced either by an equivalent force or a couple.

Proof. Equation (2.17) can be satisfied in three ways:

(a) If $\boldsymbol{F} = \boldsymbol{0}$, only the couple remains.

(b) If $\boldsymbol{M}_P = \boldsymbol{0}$, only the force remains.

(c) If both $\boldsymbol{F}$ and $\boldsymbol{M}_P$ are non-null vectors, $\boldsymbol{F}$ and $\boldsymbol{M}_P$ are perpendicular to each other.

Therefore, the couple and $\boldsymbol{F}$ lie in the same plane. This plane is shown in figure 2.42(a). Since a couple is specified fully by the provision of its moment, we can regard the couple as system $\{\boldsymbol{K}, -\boldsymbol{K}\}$ with a distance d between the lines of action of the two forces such that $Kd = M_P$. By means of the operations in §2.9, we

(a)

(b)

Figure 2.42

can now replace the couple by an equivalent couple $\{\boldsymbol{F}, -\boldsymbol{F}\}$ with a distance $\frac{K}{F}d$ between the lines of action of the forces and where $-\boldsymbol{F}$ acts at P [Figure 2.42(b)]. The two forces at P then sum to the null force and we are left with a resultant that consists of a *single force* $\boldsymbol{F}$ at Q. We note that the conclusion that we drew in §2.7.4 about two-dimensional systems of forces is also supported by the above argument. In the case of (c) above, it is of course impractical to obtain the point of application Q of the resultant each time by means of a construction. We shall rather use the fact mentioned at the end of §2.10.1, that is, that the *moment of two equivalent systems of forces about any point is the same.* □

Examples

V2.10.2. In V2.10.1 we have that $\boldsymbol{F} \cdot \boldsymbol{M}_O = 0$. Find the resultant of the system.

Solution: In V2.10.1 we reduced a given system of forces to a force $\boldsymbol{F} = 15P\hat{z}$ at O and a couple with moment $\boldsymbol{M}_O = 18Pa(-\hat{x} + \hat{y})$ [Figure 2.43(a)]. Since

(a)

(b)

Figure 2.43

$\boldsymbol{F} \neq \boldsymbol{0}$ and $\boldsymbol{M}_O \neq \boldsymbol{0}$, the resultant is a *single force* $\boldsymbol{F}$ with point of application Q [Figure 2.43(a)]. Of course, we must find the position of Q in order to specify the resultant completely. To do this, we make use of the fact that the two systems in Figures 2.43(a) and (b) are equivalent systems and that the moment of both systems about O, for instance, must be the same. It then follows that:

$$\boldsymbol{0} + \boldsymbol{M}_O = \boldsymbol{s} \times \boldsymbol{F}$$

In component form:

$$18Pa(-\hat{x}+\hat{y}) = (x\hat{x}+y\hat{y}+z\hat{z})\times(-15P\hat{z})$$
$$= 15P(x\hat{y}-y\hat{x})$$

If we now equate the x and y components on both sides of the equation, it follows that $x = \frac{6}{5}a$ and $y = \frac{6}{5}a$. We note that no restrictions are placed on z by this solution, a fact that is not surprising since $(\frac{6}{5}a, \frac{6}{5}a, z)$ are the coordinates of a *straight line* that is parallel to the Z axis and intersect the XY plane at point $(\frac{6}{5}a, \frac{6}{5}a, 0)$. Obviously, this straight line is the *line of action* of the resultant.

Problems

P2.10.1. The following forces are applied to a rigid body: $10K(\hat{z}-\hat{x})$ at $(a, 0, 0)$; $10K(\hat{x}-\hat{y})$ at $(0, a, 0)$, and $10K(\hat{y}-\hat{z})$ at $(0, 0, a)$. Find their resultant.

P2.10.2. Three forces $2K\hat{x}$, $3K\hat{y}$ and $4K\hat{z}$ are applied to the same vertex O of a cube (length of sides a) and along its edges. This system of forces is equivalent to a couple with moment $\boldsymbol{M}_P$ and a force $\boldsymbol{F}$ that is applied at point P, which is diagonally opposed to O. Calculate the magnitude and direction cosines of $\boldsymbol{F}$ and $\boldsymbol{M}_P$.

P2.10.3. Calculate the distance between A and the line of action of the resultant of the three forces in Figure 2.44 for

(a) $a = 1$ m

(b) $a = 1.5$ m and

(c) $a = 2.5$ m.

P2.10.4. The magnitude of force $\boldsymbol{F}$ in Figure 2.45 is $F = 250$ N. Replace $\boldsymbol{F}$ with an equivalent system that consists of a force at A and a couple.

P2.10.5. Calculate the resultant of the two couples that act on the crank in Figure 2.46.

Figure 2.44

Figure 2.45

P2.10.6. In Figure 2.47, two cables exert a force of 90 kN each on a truss of weight $W = 200$ kN. Calculate the resultant of the three forces.

P2.10.7. Replace the system in Figure 2.48 with a force at A and a couple.

P2.10.8. In Figure 2.49, two forces are applied to a mast. Find an equivalent system that consists of a force at O and a couple.

2.11 Equilibrium of a Rigid Body

2.11.1 General Conditions of Equilibrium

For a body to be in equilibrium, the *resultant* of the forces being exerted on it must of necessity be a *null force*. If the reduction theorem (2.16) is applied to

Figure 2.46

Figure 2.47

these forces, both the force and the moment of the couple that are obtained must therefore be null vectors. For a body that is subjected to the system in Figure 2.40, we obtain the following two *conditions of equilibrium*:

$$\begin{aligned} \boldsymbol{F} &= \sum_{i=1}^{n} \boldsymbol{F}_i = \boldsymbol{0} \\ \boldsymbol{M}_P &= \sum_{i=1}^{n} \boldsymbol{s}_i \times \boldsymbol{F}_i = \boldsymbol{0} \end{aligned} \tag{2.18}$$

A rigid body is therefore in equilibrium if:

(a) *the sum of the forces on it is zero*, and

(b) *the total moment of these forces about* any *point is zero.*

Figure 2.48

Figure 2.49

The two vector equations in (2.18) in general yield *six* scalar equations. Since a rigid body has six degrees of freedom (the minimum number of coordinates needed to specify its position and orientation), (2.18) is both *necessary* and *sufficient* for the equilibrium.

2.11.2 Equilibrium in Two Dimensions

In many applications of (2.18), all the forces being exerted on a body lie in a *plane*. Without any loss of generality, we can use a reference system that is such that the forces lie in the XY plane (Figure 2.50). The z components of the forces are then all identically equal to zero, whereas their moments about any point *in the plane* will have only z components. The six scalar equations in (2.18) are thus reduced

Figure 2.50

to *three*:

$$F_x = \sum_{i=1}^{n} F_{xi} = 0$$
$$F_y = \sum_{i=1}^{n} F_{yi} = 0 \qquad (2.19)$$
$$M_P = \sum_{i=1}^{n} M_{zi} = 0$$

In (2.19), P is *any* point in the plane, F_{xi} and F_{yi} are the x and y components of $\boldsymbol{F}_i$ repectively, $\boldsymbol{M}_{zi} = (\boldsymbol{s}_i \times \boldsymbol{F}_i) \cdot \hat{z}$ is the z component of F_i's moment about P and M_P is the z component of the total moment about O. In order to calculate M_{zi}, it is only necessary to calculate the magnitude of $\boldsymbol{F}_i$'s moment about P by using (2.11) and then to allocate a positive (negative) sign to it if, for the reference system in Figure 2.50, the moment has an anti-clockwise (clockwise) sense.

Examples

We shall accept the following fact as a given throughout: the weight of a symmetrical, uniform body acts at its point of symmetry.

V2.11.1. A uniform rod, length $16a$, rests in equilibrium against a smooth wall and on a smooth nail at a distance a from the wall. Calculate the angle between the rod and the wall.

Solution: The forces acting on the rod (Figure 2.51) are its weight $\boldsymbol{W}$ (which acts at its point of symmetry B) and the reaction forces $\mathbf{N}$ and $\boldsymbol{R}$. Since there is

Figure 2.51

no friction, the latter two forces are *perpendicular* to the contact surfaces on which they are exerted. For the selected reference system, we now write down the three conditions of equilibrium (2.19). In order to calculate the total moment of the system, a point of reference P is chosen such that it reduces algebraic calculations to a minimum, usually by making it lie on the line of action of one or more of the unknown forces.

$$F_x = 0: \quad R - N\cos\alpha = 0 \quad \text{(a)}$$
$$F_y = 0: \quad N\sin\alpha - W = 0 \quad \text{(b)}$$
$$M_P = 0: \quad N\left(\frac{a}{\sin\alpha}\right) - W(8a\sin\alpha) = 0 \quad \text{(c)}$$

We note that we have three equations in the three unknowns R, N and α, and the problem is therefore well defined.

It follows then from (b) that

$$N = \frac{W}{\sin\alpha},$$

and if we substitute this value for N in (c), we obtain an equation in which α is the only unknown:

$$W\left(\frac{a}{\sin^2\alpha}\right) - 8Wa\sin\alpha = 0.$$

It follows immediately that

$$\sin\alpha = \frac{1}{2}, \quad \alpha = 30^\circ.$$

Equations (b) and (a) can now be used to calculate N and R.

V2.11.2. A uniform ladder rests with its ends respectively on a rough floor (friction coefficient μ) and a smooth wall. Find the slope angle of the ladder if it is just about to start sliding.

Figure 2.52

Solution: The forces being exerted on the ladder are shown in Figure 2.52. Since the ladder is about to slide, $f = \mu N$ and $\boldsymbol{f}$ is directed such that it *opposes* the imminent movement of the ladder. It follows from (2.19) that

$$\begin{aligned} F_x = 0: &\quad R - \mu N = 0 &\quad \text{(a)} \\ F_y = 0: &\quad N - W = 0 &\quad \text{(b)} \\ M_A = 0: &\quad W a\cos\alpha - R(2a\sin\alpha) = 0 &\quad \text{(c)} \end{aligned}$$

Once again we have three equations in the three unknowns R, N and α. It then follows from (b) that

$$N = W \tag{d}$$

and (d) can be substituted into (a) to obtain

$$R = \mu W \tag{e}$$

If (e) is now substituted in (c), we obtain an equation in which α is the only unknown,

$$Wa\cos\alpha - 2\mu Wa\sin\alpha = 0,$$

after which we can solve for α:

$$\tan\alpha = \frac{1}{2\mu}$$

V2.11.3. A sphere, radius a and weight W, rests on a smooth surface that forms an angle of 60° with the horizontal. A rope of length a, the one point of which is tied to the surface of the sphere and the other point of which is tied to the inclined surface, holds the sphere in position. Calculate the angle that the rope forms with the inclined surface as well as the tension in the rope.

Figure 2.53

Solution: The three forces being exerted on the sphere are shown in Figure 2.53. Weight $\boldsymbol{W}$ acts at the point of symmetry O of the sphere, and we know from elementary geometry that the line of action of the normal reaction force $\boldsymbol{N}$ also passes through O. It follows immdiately from the last condition of equilibrium in (2.19), $M_O = 0$, that the line of action of force $\boldsymbol{T}$ in the rope must also pass through O. From the geometry of $\triangle OAB$, we then have:

$$\sin\alpha = \frac{a}{2a} = \frac{1}{2}$$

and therefore

$$\alpha = 30^\circ.$$

We obtain the magnitude of the force in the rope directly from the first condition in (2.18):

$$F_x = 0 : \qquad T\cos\alpha - W\sin 60^\circ = 0$$

It then follows that

$$T = W.$$

V2.11.4. The uniform concrete beam (weight 2.5×10^5 N) shown in Figure 2.54 is lowered slowly with the help of two cables, each of which can yield a maximum tensile force of 5×10^5 N.

Figure 2.54

(a) Will the cables resist the weight, and if not, which one will break first?

(b) If a cable breaks, at which stage will this happen?

(Given: $AB = 6$ m, $AC = 6$ m, $CE = 2$ m).

Solution: Let $a = 2$ m and $W = 2.5 \times 10^5$ N in Figure 2.54. The forces being exerted on the beam are its weight $\boldsymbol{W}$, forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ in the cables and the normal reaction force $\boldsymbol{N}$ at A. We consider the equilibrium of the beam when it forms an angle θ with the horizontal. We need to write down only two of the conditions of equilibrium (2.19):

$$F_x = 0: \qquad T_2 - T_1 \cos\frac{\theta}{2} = 0 \qquad \text{(i)}$$

$$M_A = 0: \quad T_1(3a \sin\frac{\theta}{2}) - W(2a\cos\theta) = 0 \qquad \text{(ii)}$$

From (i) we have

$$T_2 = T_1 \cos\frac{\theta}{2}.$$

Since

$$\cos\frac{\theta}{2} \leq 1 \quad \text{for} \quad 0° \leq \theta \leq 180°,$$

it follows that $T_1 \geq T_2$ and cable BC will be the first to break. The cable will break if there is a value of θ for which $T_1 = 5 \times 10^5 = 2W$. To calculate whether or not there is such a value of θ, we put $T_1 = 2W$ in (ii):

$$6Wa \sin\frac{\theta}{2} - 2Wa\cos\theta = 0$$

We then establish whether or not this equation can be solved for θ. We first write the whole equation in terms of $\frac{\theta}{2}$.

$$3\sin\frac{\theta}{2} - (1 - 2\sin^2\frac{\theta}{2}) = 0$$

We then obtain a quadratic equation in $\sin\frac{\theta}{2}$,

$$2\sin^2\frac{\theta}{2} + 3\sin\frac{\theta}{2} - 1 = 0,$$

with solutions

$$\sin\frac{\theta}{2} = \frac{-3 \pm \sqrt{17}}{4} = 0.281 \quad \text{or} \quad -1.781.$$

Since $\left|\sin\frac{\theta}{2}\right| \leq 1$ the second solution is not valid. However, the first solution determines the value of θ for which cable BC will break. We can now answer the whole question:

(a) Cable BC will break first, for

(b) $\theta = 32.61°$.

Problems

P 2.11.1. A uniform rod of length $2a$ is placed in a rough, hollow sphere with radius $\sqrt{2}a$. If the rod is just in equilibrium in a vertical plane through the centre of the sphere, show that it forms an angle of 2λ with the horizontal, where λ is the angle of friction at both end points.

P2.11.2. A sphere, radius a and weight W, rests against a smooth vertical wall and is held in position by a rope, length $(\sqrt{2}-1)a$, attached with the one end to the wall and the other end to the surface of the sphere. Calculate the angle that the rope forms with the wall as well as the tension in the rope.

P2.11.3. A uniform rod (weight W) can rotate freely around one end while the rod is held in position at an angle of $60°$ with the upward vertical by a rope that forms an angle of $45°$ with the rod and is in the same vertical plane as the rod. Calculate the tension in the rope and the reaction at the joint.

P2.11.4. Rod AB, weight W, can rotate freely around a hinge in a wall at A and is further supported by a horizontal rope BC that is tied to a point C in the wall. Given that $BC = 4a$ and $AC = a$, calculate force T in the rope and the reaction R at the hinge.

P2.11.5. A uniform rod rests with one end on a rough horizontal surface, friction coefficient μ, and the other end is supported by a force that is exerted perpendicularly to the rod. Prove that if α is the maximum angle of inclination of the rod just before it starts sliding, α is given by the equation $2\mu\tan^2\alpha - \tan\alpha + \mu = 0$. What can you deduce for the case $\mu = \frac{1}{2}$?

P2.11.6. A uniform ladder rests with one end on a horizontal floor and with the other end against a vertical wall. The friction coefficients are $\frac{2}{5}$ and $\frac{1}{2}$ respectively. Calculate the angle of inclination of the ladder if it is about to start sliding.

P2.11.7. A uniform rod AB, length ℓ and weight W, which forms an angle of $30°$ with the upward vertical, rests with one end A against a smooth vertical wall. A rope that is attached to a point C on the rod with $AC = \frac{\ell}{4}$ keeps it in position. The other end of the rope is tied to a point D vertically above A. Calculate the angle that the rope forms with the vertical as well as the force in the rope.

P2.11.8. A thin uniform rod rests in equilibrium with one end point on a smooth inclined surface, with angle of inclination $60°$, and the other end on a rough horizontal surface. The vertical plane through the rod intersects the inclined plane in a line of largest inclination. If the rod is about to slide when it forms an angle of $30°$ with the horizontal, prove that the friction coefficient between the rod and surface is $\frac{1}{\sqrt{3}}$.

P2.11.9. A uniform ladder, length $13a$ and weight W, rests with one end against a smooth vertical wall at a height $12a$ above a smooth horizontal floor and with its other end on the floor. The ladder is prevented from sliding by means of a horizontal rope, the one end of which is tied to the bottom of the ladder and the other end to the wall. A man weighing $9W$ climbs up the ladder carefully, but the rope breaks when he reaches the middle rung of the ladder. What is the maximum tensile force that can be supplied by the rope?

P2.11.10. The hoop in Figure 2.55, weight W and diameter $4a$, is held in equilibrium by two smooth nails lying in a vertical line. What force is exerted on each nail?

Figure 2.55

P2.11.11. The uniform rod AB in Figure 2.56 lies in a vertical plane with its end point against the smooth surfaces AC and BC. Calculate the angle θ for equilibrium if

(a) $\alpha = 30°$

(b) $\alpha = 40°$ and

(c) $\alpha = 60°$.

Figure 2.56

P2.11.12. A garage door (weight 900 N) is shown in Figure 2.57. It consists of a uniform right-angled panel AC, 2.4 m high, supported by cable AE, which is tied to the centre point of the topmost edge of the door and by two sets of frictionless rollers at A and B. Each set consists of two rollers on opposite sides of the door. The rollers A are free to move in horizontal grooves and the rollers B move in vertical grooves. If the door is held in a position where $BD = 1.2$ m, calculate:

(a) the tension in cable AE and

(b) the reaction on each of the four rollers.

P2.11.13. A traffic light can be supported in three ways, as shown in Figure 2.58. In the case of (c), the tension in cable BC is measured as 1950 N. Calculate the reactions at A for each of the three cases.

Figure 2.57

(a) (b) (c)

Figure 2.58

P2.11.14. While the engine of the aircraft in Figure 2.59 is running, the vertical reaction that the ground exerts on wheel A is measured to be 5 kN. If the engine is switched off, the vertical reactions at A and B are measured to be 3.75 kN each. The difference in the readings at A is caused by a couple that is exerted on the screw when the engine is running. This couple tends to turn the aircraft anti-clockwise, which opposes the clockwise rotation of the screw. Calculate the magnitude of this couple and the magnitude of the vertical force exerted on B when the engine is running.

P 2.11.15. Calculate the magnitude and direction θ of the bone force $\boldsymbol{F}_B$ and the magnitude of the muscle force $\boldsymbol{T}$ required to maintain a ball of weight 100 N

Figure 2.59

(Figure 2.60) in equilibrium. The weight of the ball acts at G.

Figure 2.60

P2.11.16. The weight of the missile in Figure 2.61 is 15 kN and is exerted at G. It is lifted to the launching position by means of a hydraulic cylinder that is housed in arm AB. If the launching pad CD has a weight of 8 kN, which is exerted at G', calculate the force that develops in AB when the pad is in the indicated position.

P2.11.17. The weight of the self-propelled crane AB in Figure 2.62 is 500 kN and is exerted at G. Calculate the weight of the heaviest mass that can be suspended at C without tipping the crane.

Figure 2.61

Figure 2.62

2.11.3 Equilibrium in Three dimensions

Generally, if forces are directed in arbitrary directions, the complete set of conditions in (2.18) must be enforced. This implies that all the $\boldsymbol{s}_i$ and $\boldsymbol{F}_i$ must be written in component form and that al the components of the two vector equations must be set equal to zero.

Examples

V2.11.5. Sphere E (weight W) in Figure 2.63 is supported by a mass that rests on a ball-and-socket joint and is supported by cables AB and AC. The weight of the mass and the cables is negligible in comparison with that of the sphere. Calculate the reaction at D and the forces in the cables.

Figure 2.63

Solution: The first condition of equilibrium in (2.18) is as follows in this case:

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + \boldsymbol{R} + \boldsymbol{W} = \boldsymbol{0} \tag{a}$$

In (a) $\boldsymbol{W}$ is a force, the magnitude and direction of which are known:

$$\boldsymbol{W} = W(-\hat{z}) \tag{b}$$

The reaction $\boldsymbol{R}$ in the hinge is a force, the point of application of which is known (D), but the magnitude and direction of which are unknown. Therefore, we can regard this as a force with three unknown components:

$$\boldsymbol{R} = R_x\hat{x} + R_y\hat{y} + R_z\hat{z} \tag{c}$$

The magnitude T_1 of force $\boldsymbol{T}_1$ in cable AC is unknown, but its direction is known, that is, it is the same as that of line segment $\overline{AC}$. Therefore, we can write that

$$\boldsymbol{T}_1 = \boldsymbol{T}_1\hat{n}_1 = T_1\frac{\overline{AC}}{|\overline{AC}|}.$$

For $A(0,0,10)$ and $C(-4,-\frac{1}{2},0)$ it follows that $\overline{AC} = -4\hat{x} - \frac{1}{2}\hat{y} - 10\hat{z}$ and $\left|\overline{AC}\right| = \frac{1}{2}\sqrt{465}$, and we have

$$\boldsymbol{T}_1 = \frac{T_1}{\sqrt{465}}(-8\hat{x} - \hat{y} - 20\hat{z}). \tag{d}$$

Similarly, we have the following for the force in cable AB that

$$\boldsymbol{T}_2 = T_2 \hat{n}_2 = T_2 \frac{\overline{AB}}{|\overline{AB}|}.$$

For $A(0,0,10)$ and $B(-\frac{1}{2},-\frac{5}{2},0)$ it follows that $\overline{AB} = -\frac{1}{2}\hat{x} - \frac{5}{2}\hat{y} - 10\hat{z}$ and $|\overline{AC}| = \frac{1}{2}\sqrt{426}$, and therefore

$$\boldsymbol{T}_2 = \frac{T_2}{\sqrt{426}}(-\hat{x} - 5\hat{y} - 20\hat{z}) \qquad \text{(e)}$$

Equations (b) to (e) are now substituted in (a) and similar components are collected:

$$\left(-\frac{8T_1}{\sqrt{465}} - \frac{T_2}{\sqrt{426}} + R_x\right)\hat{x} + \left(-\frac{T_1}{\sqrt{465}} - \frac{5T_2}{\sqrt{426}} + R_y\right)\hat{y} + \left(-\frac{20T_1}{\sqrt{465}} - \frac{20T_2}{\sqrt{426}} + R_z - W\right)\hat{z} = \mathbf{0}$$

Since each component of the vector on the left hand side of the above equation must be zero, we have the three scalar equations

$$R_x = \frac{8T_1}{\sqrt{465}} + \frac{T_2}{\sqrt{426}} \qquad \text{(f)}$$

$$R_y = \frac{T_1}{\sqrt{465}} + \frac{5T_2}{\sqrt{426}} \qquad \text{(g)}$$

$$R_z = \frac{20T_1}{\sqrt{465}} + \frac{20T_2}{\sqrt{426}} + W \qquad \text{(h)}$$

in the five unknowns R_x, R_y, R_z, T_1 and T_2. The other two equations required to solve this problem are obtained from the second condition in (2.18):

$$\boldsymbol{M}_D = \mathbf{0}$$

D is chosen as a reference point because $\boldsymbol{R}$ does not have a moment about it. The three unknowns R_x, R_y and R_z will therefore not occur in this equation of equilibrium. We now calculate the total moment about D:

$$\begin{aligned}
\boldsymbol{M}_D &= \boldsymbol{r}_A \times \boldsymbol{T}_1 + \boldsymbol{r}_A \times \boldsymbol{T}_2 + \boldsymbol{r}_E \times \boldsymbol{W} \\
&= 10\hat{z} \times \frac{T_1}{\sqrt{465}}(-8\hat{x} - \hat{y} - 20\hat{z}) + 10\hat{z} \times \frac{T_2}{\sqrt{426}}(-\hat{x} - 5\hat{y} - 20\hat{z}) \\
&\quad + (2\hat{x} + \frac{5}{2}\hat{y} + \hat{z}) \times (-W\hat{z}) \\
&= \frac{10T_1}{\sqrt{465}}(-8\hat{y} + \hat{x}) + \frac{10T_2}{\sqrt{426}}(-\hat{y} + 5\hat{x}) + W(2\hat{y} - \frac{5}{2}\hat{x}) \\
&= (\frac{10T_1}{\sqrt{465}} + \frac{50T_2}{\sqrt{426}} - \frac{5}{2}W)\hat{x} + (-\frac{80T_1}{\sqrt{465}} - \frac{10T_2}{\sqrt{426}} + 2W)\hat{y}
\end{aligned}$$

Since all the components of $\boldsymbol{M}_D$ must be zero, we obtain the two scalar equations required to obtain the solution:

$$\frac{10T_1}{\sqrt{465}} + \frac{50T_2}{\sqrt{426}} = \frac{5}{2}W \qquad \text{(i)}$$

$$\frac{80T_1}{\sqrt{465}} + \frac{10T_2}{\sqrt{426}} = 2W \qquad \text{(j)}$$

The last two equations can be solved immediately for the forces in the cables,

$$\begin{aligned}
T_1 &= \frac{\sqrt{465}}{52}W = 0.415W \\
T_2 &= \frac{3\sqrt{426}}{65}W = 0.953W,
\end{aligned}$$

and these values can be substituted in the right hand side of equations (f), (g) and (h) to find the components of the reaction on the joint:

$$R_x = \frac{W}{5}, \quad R_y = \frac{W}{4}, \quad R_z = \frac{30}{13}W.$$

The magnitude of the reaction force is therefore $R = 2.330W$, and is a *compressive force.*

Question: Why are there only *five* unknowns in this problem?

Problems

P2.11.18. A force of 8.4 kN is exerted on a 10 m pole, as shown in Figure 2.64. The pole is supported by a ball-and-socket hinge at A and by two cables BD and BE. The weight of the pole is negligible. Calculate the forces in the two cables as well as the reaction at A.

Figure 2.64

P2.11.19. Beam AC in Figure 2.65 is supported by a ball-and-socket joint at A and by two cables BDC and CE. Cable BDC moves over a frictionless pulley at D. Calculate the forces in the cables and the reaction at A if a mass of weight W is attached to the beam at C.

Figure 2.65

P2.11.20. The rigid L-beam in Figure 2.66 is supported by a ball-and-socket joint at A and the three cables DB, EB and FC. Calculate the force in each cable and the reaction at A if the beam is loaded with a weight of 5 kN at G.

Figure 2.66