Chapter 1

Vector Algebra

1.1 Introduction

Vector algebra and vector analysis were developed in the nineteenth century and has—in the past 50 years—become a necessary part of the mathematical background of mathematicians, physicists, engineers and other scientists. The reasons for this are that these disciplines allow for the *compact reformulation* of the mathematics associated with geometric and physical problems and facilitate the formation of *intuitive images* of physical and mathematical concepts. The "vector language" is in fact the most natural way of thinking in the physical sciences, and students are encouraged to make this way of thinking their own.

This chapter develops vector algebra on the basis of simple geometric concepts already known to the students. The algebraic rules that are developed in this way are applied mainly to geometric problems by way of illustration. In chapter 2 the concepts of statics will then be developed in a comprehensive vector algebraic manner.

1.2 Reference Systems

A concept that is very important in mechanics is that of the *particle*: a body, which in a given situation, can be regarded as small enough to be described as a mass at a *geometrical point.* Therefore, it is necessary to be able to establish the position of a geometrical point. To this end we make use of a *fixed reference system*. Such a system is set up in a way that makes it possible for *any* point in a relevant space to be labeled *uniquely* by means of a *minimal* set of numbers.

1.2.1 One-dimensional Reference System

As an example, consider a bead that moves on a wire (Figure 1.1). In this instance,

Figure 1.1

a reference system can, *for example*, be introduced as follows: a *fixed* point O on the wire is chosen to serve as the so-called *origin* of the system. A choice is also made regarding the direction in which the distances along the wire will be regarded as *positive* distances. Note the convention that is used throughout in the sketches in these notes: the positive direction of an axis of reference is always indicated with an arrow. Then the distance s (with the correct *sign*), measured *along the wire*, establishes P's position uniquely. In this case, we will refer to s as the ***s** coordinate* of the relevant point.

However, we realise immediately that the $O'X$ system can also serve as a reference system for the relevant problem: the x coordinate is the distance (with the correct sign) between the origin O' and the *perpendicular projection* of P on the X axis. Clearly, x is unique for a given position of P and *vice versa*.

The dimension (number of coordinates) of a reference system is unique; not the choice of system.

1.2.2 Two-dimensional Reference System

Consider a particle that moves in a plane surface (Figure 1.2). Here we can, *for*

Figure 1.2

example, use *two* mutually orthogonal, straight axes of reference through a fixed origin O. Point P's position can then be determined by means of the choice of the *pair* of numbers (x, y), where x and y are the distances (with the correct signs) between the origin and the projections of P on the x and y axes respectively.

NB: A plane surface is not the only possibility of a two-dimensional space—think about the surface of the earth, for instance.

1.2.3 Three-dimensional Reference System

This requires (for example) *three* mutually orthogonal axes of reference through a *fixed origin* O (Figure 1.3). Note the way in which the three-dimensional figure is *represented* on the two-dimensional surface of the page: the X and Y axes are drawn to lie *in* the plane surface of the page. They are then drawn perpendicular to each other and respectively parallel to the borders of the page. The Z axis, which is *normal* to the page, is drawn so that its *positive* direction forms *obtuse* angles with the *positive* directions of both the X and Y axes.

In keeping with the general convention, we will always use a *right-hand system:* if the spiral part of a corkscrew with a *right-handed thread* is held *perpendicular* to the XY plane and the handle is turned along the *shortest* route of the *positive* X direction to the *positive* Y direction, the spiral part moves in the *positive* Z

Figure 1.3

direction. It is easily established that this rule is *consistent*: if the permutations $X \to Y \to Z \to X$ are made in the previous sentence, the sentence will still be true.

A reference system, as described above, is called a *rectangular right-handed system* or a *Cartesian system*. Figure 1.4 illustrate how the coordinates of a point with regards to such a reference system are obtained. The x (y, z) coordinate of P

Figure 1.4

is determined by constructing a plane surface that contains P and is *perpendicular* to the X (Y, Z) axis. The distance between the origin and the point where the plane cuts the X (Y, Z) axis, with the correct sign, is then the x (y, z) coordinate of P. We will indicate the coordinate of point P with the notation $P(x, y, z)$. (Note another convention that is used in drawings of three-dimensional figures: in Figure 1.4, all the lines that would have been invisible if the planes in the figure

Figure 1.5

were non-transparent are indicated by dotted lines.)

Problems

P1.2.1.

Draw sketches for the ***same*** choice of a reference system which shows the position of the points with coordinates $(1, 1, 1)$; $(1, -1, 1)$; $(1, 1, -1)$ and $(-1, -1, -1)$.

1.3 Displacement

The concept of displacement is fundamental to the description of motion. If a particle moves from point $P_1(x_1, y_1, z_1,)$ to point $P_2(x_2, y_2, z_2)$, the *nett* change in its position is determined uniquely by the *initial point* P_1 and the *end point* P_2. The displacement of a particle is therefore characterised completely by the *ordered number pair* P_1P_2. It is natural to *represent* this displacement on a sketch by means of a *directed line segment*; see Figure 1.5.

1.3.1 Notation

It is evident that the directed line segment that represents the displacement between P_1 and P_2 contains both a magnitude (measured in metres) and a direction. For this reason, we will differentiate between a displacement and other quantities, the magnitudes of which are the only important aspect, by means of notation. The displacement between P_1 and P_2 is indicated by means of a bar as follows: $\overline{P_1P_2}$ (note the order of P_1 and P_2). As indicated in Figure 1.5, we will also indicate the displacement with a single letter, and this letter is then differentiated from other letters in the text. In hand-written text and in these notes, we use a bar over the letter, for example $\overline{a}$. Other notations in the literature include the use of bold face symbols, for example, $\boldsymbol{a}$, or $\overrightarrow{P_1P_2}$, $\vec{a}$, $\tilde{a}$, $\underline{a}$. If we refer only to the *magnitude* of the displacement in Figure 1.5, we will indicate it by $\left|\overline{P_1P_2}\right|$, $|\overline{a}|$, or a.

1.3.2 Magnitude and direction of displacement

Since the displacement $\overline{P_1P_2}$ is determined uniquely by the position of P_1 and P_2, we must be able to *calculate* both the magnitude and direction of the displacement from the *coordinates* of the two points. For this purpose, we first define the x, y and z *components* respectively, as follows:

$$a_x := x_2 - x_1, \quad a_y := y_2 - y_1, \quad a_z := z_2 - z_1. \tag{1.1a}$$

We note the *order* in the equation (1.1a): the coordinates of the *initial point* are deducted from those of the *end point*.

The geometric significance of the displacement's components is clear in Figure 1.6(a): the components of $\overline{P_1P_2}$ are the side lengths of a rectangular volume which has the displacement $\overline{P_1P_2}$ as its *diagonal*. The *magnitude* of the displacement, that is, the straight line distance between P_1 and P_2, can now be obtained with the help of Pythagoras' theorem. Firstly, it follows for $\triangle P_1QT$ that

$$c^2 = a_x^2 + a_y^2,$$

and then for $\triangle P_1TP_2$:

$$a^2 = c^2 + a_z^2 = a_x^2 + a_y^2 + a_z^2$$

The *magnitude* of displacement $\overline{P_1P_2}$ is therefore

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{1.1b}$$

Note that we take the *positive* root in (1.1b).

(a)

(b)

Figure 1.6

A natural way of determining the *direction* of $\overline{P_1P_2}$ is shown in Figure 1.6(b). Angles α, β and γ are the angles between the *forward* direction of $\overline{P_1P_2}$ and the *positive* directions of the x, y and z axes respectively. These angles can be obtained from the so-called *direction cosines* ℓ, m and n, which can be obtained directly from the inspection of right-angled triangles $\triangle P_1QP_2$, $\triangle P_1RP_2$ and $\triangle P_1SP_2$:

$$\begin{aligned} \ell &:= \cos\alpha = \frac{a_x}{a} \\ m &:= \cos\beta = \frac{a_y}{a} \\ n &:= \cos\gamma = \frac{a_z}{a} \end{aligned} \tag{1.1c}$$

If we square both sides of (1.1c) and calculate the sum of all three equations, we

Figure 1.7

obtain a very useful identity:

$$\ell^2 + m^2 + n^2 = 1. \tag{1.1d}$$

Note that (1.1d) implies that α, β and γ are not independent of each other.

1.3.3 Composition of displacements

Figure 1.7 shows two *successive* displacements: displacement $\overline{P_1P_2}$ followed by displacement $\overline{P_2P_3}$. It is clear that these two displacements together have the same *nett* effect as *one* displacement, viz. $\overline{P_1P_3}$. It is natural to express this equivalence as follows:

$$\overline{P_1P_3} = \overline{P_1P_2} + \overline{P_2P_3}. \tag{1.2}$$

1.3.4 Dimensions

Unless otherwise indicated, *distance* will always be given in the SI unit, viz. metre.

Problems

P1.3.1. P and Q are points with coordinates $(3, -2, 1)$ and $(-1, 1, 1)$ respectively.

(a) What is the distance from P to Q?

(b) Which angles does the directed line segment $\overline{PQ}$ make with the positive direction of the axes of coordinates?

P1.3.2. A helicopter flies 100 metres vertically upwards, then 700 metres horizontally south and then 300 metres horizontally east. How far is the helicopter from its starting point?

P 1.3.3. $ABCD$ is the floor and $EFGH$ the ceiling of a cubic room; all the measurements are 4 metres and E, F, G and H lie vertically above A, B, C and D respectively. L is a point in the centre of the ceiling and J is the centre point of the floor. Choose the positive directions of the X, Y and Z axes to coincide with the forward direction of directed line segments $\overline{AB}$, $\overline{AD}$ and $\overline{AE}$ respectively. Carry out the following for each of the directed line segments $\overline{LD}$, $\overline{JH}$, $\overline{DF}$, $\overline{HF}$, $\overline{AE}$, $\overline{CB}$, $\overline{CD}$ and $\overline{AG}$:

1. Name the directed line segments on the floor and the ceiling, which are the projections of the above line segments.

2. Calculate the lengths and direction cosines of the directed line segments with regard to the XYZ system.

3. Calculate the angles that the line segments form with the X, Y and Z axes.

P1.3.4. In problem 3 there is a spider at A that notices an ant at point M on the ceiling; the angle of elevation of the ant at A is θ where $\cos\theta = \frac{3}{\sqrt{17}}$ and M lies vertically above a line that forms an angle of 45° with the X axis.

1. Calculate

   (a) The coordinates of M.

   (b) The distance between A and M.

   (c) The angles that $\overline{AM}$ forms with the positive direction of the coordinates.

2. The spider could use a large number of directed line segments to reach the ant, for example, $\overline{AD} + \overline{DH} + \overline{HM}$.

   (a) List four more possibilities.

(b) Which of these possibilities will give the spider the *shortest* distance to the ant?

(c) Is this the *smallest* of all possiblities? If not, find the directed line segments that represent the shortest distance if the spider maintains constant contact with the floor, walls and ceiling.

P1.3.5. Point $P(2, 5, d)$ is a distance of 5 metres from $Q(-1, 1, 4)$. Find d.

P1.3.6. The positive X axis is horisontal and directed to the east, the positive Y axis is vertical and directed upwards and the positive Z axis is horizontal and directed to the south. An observer sits on the ground at the origin. The top-most point of the church steeple has an angle of elevation of $30°$ with regard to the origin and lies in a direction $50°$ east of south. Which angles does the line from the observer to the steeple's top-most point form with the coordinate axes?

P1.3.7. O is the origin and P is another point. Segment OP forms an angle of $60°$ with the positive X axis and an angle of $45°$ with the positive Y axis. What could be said about the angle that it forms with the positive Z axis?

1.4 Vectors

We saw in §1.3 that displacement has both magnitude and direction. Many other entities of the same nature are found in the physical sciences, for example, force, velocity, acceleration, electrical field and magnetic field. As with displacement, it is convenient to represent these quantities on a chosen *scale* by means of a *directed* line segment or *arrow*. We refer to these quantities as *vectors*.

1.4.1 Definition

A *vector* is a quantity that can be *represented* by a directed line segment or arrow, which has specific *magnitude and direction*.

Consider for instance arrow OP in Figure 1.8, drawn from O in a NW direction. It is 4cm long. This arrow can now be used to represent:

(a) A displacement of 400 kilometres in a NW direction on a scale of 1cm : 100km.

(b) A force of 16 Newton in a NW direction on a scale of 1cm : 4 Newton.

(c) A velocity of 4 metres per second in a NW direction on a scale of 1cm : 1 metre per second.

(d) A magnetic field of 4.0×10^{-5} tesla in a NW direction on a scale of 1 cm : 10^{-5}.

Figure 1.8

1.4.2 Scalar

A *scalar* is a quantity that can be specified by a *single* number (positive, negative or zero). Examples of such quantities are time, temperature, energy and mass.

1.4.3 Notation

The same notation used in §1.3.1 for displacements will be used for vectors.

1.4.4 Localised Vectors

It is sometimes customary to represent a vector by means of three symbols (x, y, z) where x, y and z are the coordinates of the end point of the arrow representing a vector. In this case we assume implicitly that the vector is *localised*, that is, that the initial point of the arrow is at the origin. In mechanics we deal with vectors that do not necessarily take effect at the origin. Think about a group of people pushing a car; the forces (vectors) take effect where their hands and the car make contact. Therefore, we prefer to think about vectors in terms of directed line segments because we can then represent the *magnitude*, *direction* and *point of action* visually.

1.5 Algebraic rules for vectors

The algebraic rules for vectors are introduced on the basis of the *geometric properties* of displacements. From the definition of other vector quantities (for example, velocity, acceleration, momentum, force) it could then be shown with the necessary mathematical tools that these rules apply to all vector quantities.

1.5.1 Equality of vectors

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are called *equal vectors* if they have the same *magnitude* and *direction*. This means that they are represented by two arrows on the same scale with the same length and direction (Figure 1.9). We shall indicate this equality as follows:

$$\boldsymbol{a} = \boldsymbol{b}. \tag{1.3}$$

1.5.2 Negative vectors

In Figure 1.10, $\boldsymbol{a}$ and $\boldsymbol{b}$ are the same magnitude but their directions are in *opposition*. We then say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *opposites*, and we indicate this as follows:

$$\boldsymbol{a} = -\boldsymbol{b}. \tag{1.4}$$

The following is evident from this definition:

$$-(-\boldsymbol{a}) = \boldsymbol{a}. \tag{1.5}$$

Figure 1.9

Figure 1.10

1.5.3 Vector sum

The sum of two given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a third vector $\boldsymbol{c}$, which is obtained as though they were *successive displacements*. We denote the vector sum in Figure 1.11 as follows:

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}. \tag{1.6}$$

Note the order: vector $\boldsymbol{a}$ appears *first* in $\boldsymbol{a}+\boldsymbol{b}$, so it is represented *first*; vector $\boldsymbol{b}$'s representation is then drawn so that its *initial point* coincides with the end point of $\boldsymbol{a}$'s representation. The *algebraic* characteristics of the vector sum can all be obtained from the definition contained in Figure 1.11:

(a) The vector sum is *closed*—the sum $\boldsymbol{a} + \boldsymbol{b}$ is also a *vector*.

(b) The vector sum is *commutative*—it follows from elementary geometric con-

Figure 1.11

siderations in Figure 1.12 that

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}. \tag{1.7}$$

Figure 1.12

(c) The vector sum is *associative*—it follows from Figure 1.13 that

$$\boldsymbol{d} = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}). \tag{1.8}$$

Figure 1.13

1.5.4 Null vector

A displacement $\boldsymbol{a}$ followed by $-\boldsymbol{a}$ results in no displacement—this is called the *null displacement*. Similarly, the sum of any vector and its opposite produces a vector the magnitude of which is zero and the direction of which cannot be determined. We call this vector the *null vector* and indicate it by $\mathbf{0}$ ($\overline{0}$ in written work).

$$\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}. \tag{1.9}$$

If the null vector is added to any vector, the vector will obviously remain unchanged:

$$\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}. \tag{1.10}$$

1.5.5 Vector difference

The difference between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as

$$\boldsymbol{a} - \boldsymbol{b} := \boldsymbol{a} + (-\boldsymbol{b}) \tag{1.11}$$

and is illustrated graphically in Figure 1.14. In this figure it is useful to remember the definition with reference to $\triangle OAB$. It is important to note that Figure 1.14 also contains a *vector sum*:

$$(\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a}. \tag{1.12}$$

Figure 1.14

This fact, together with properties (1.7) and (1.8) of the vector sum, implies that vector expressions can be *manipulated* like *scalar expressions* with regard to the $+$ and $-$ operations.

1.5.6 Product of scalar and vector

The vector sum $\boldsymbol{a}+\boldsymbol{a}+\boldsymbol{a}$ is a vector that has the same direction as $\boldsymbol{a}$, and its magnitude is *three* times that of $\boldsymbol{a}$. It is natural to represent this sum by $3\boldsymbol{a}$. This leads us to the following definition. For $p>0$, we define $p\boldsymbol{a}$ as a *vector* as follows:

$$\begin{aligned} p\boldsymbol{a} \parallel \boldsymbol{a};\ |p\boldsymbol{a}| &= p\,|\boldsymbol{a}| \\ (-p)\boldsymbol{a} &= -(p\boldsymbol{a}) \\ 0\boldsymbol{a} &= \boldsymbol{0} \end{aligned} \tag{1.13}$$

This definition is represented visually in Figure 1.15 for $p>1$. The algebraic properties of this product once again follows from definition.

(a) *Associativity*—it follows from (1.13) that

$$p(q\boldsymbol{a}) \parallel q\boldsymbol{a} \parallel \boldsymbol{a}$$

and

$$|p(q\boldsymbol{a})| = p\,|q\boldsymbol{a}| = p\,(q\,|a|) = (pq)\,|\boldsymbol{a}|$$

Figure 1.15

and hence

$$p(q\boldsymbol{a}) = (pq)\boldsymbol{a}. \qquad (1.14)$$

(b) *Distributivity*—it follows from Figure 1.16 that since

(i) $\boldsymbol{a} \parallel p\boldsymbol{a}$,

(ii) $\boldsymbol{b} \parallel p\boldsymbol{b}$,

(iii) $\dfrac{|p\boldsymbol{a}|}{|\boldsymbol{a}|} = \dfrac{|p\boldsymbol{b}|}{|\boldsymbol{b}|} = p$,

that $\triangle OAB$ en $\triangle O'A'B'$ are similar. It then follows that:

Figure 1.16

$$\boldsymbol{c} \parallel (\boldsymbol{a}+\boldsymbol{b}); \quad \frac{|\boldsymbol{c}|}{|\boldsymbol{a}+\boldsymbol{b}|};$$

that is,

$$\boldsymbol{c} = p(\boldsymbol{a} + \boldsymbol{b})$$

and hence

$$p(\boldsymbol{a} + \boldsymbol{b}) = p\boldsymbol{a} + p\boldsymbol{b}. \tag{1.15}$$

The properties (a) and (b) imply that the product of a scalar with a vector can be manipulated in the same way as the product of two scalars.

1.6 Geometric applications

Since the definitions in §1.5 implicitly contain the properties of triangles, a great number of geometric problems pertaining to triangles can be solved in a compact way in terms of vector operations. We shall now discuss some examples.

Examples

V1.6.1. Show that if the midpoints of the successive sides of a *quadrilateral* are connected, a *parallelogram* will be obtained.

Solution: We begin by writing the given information in vector form. The fact that the line segments AB, BC, CD and DA are halved by E, F, G and H respectively is made evident in Figure 1.17 by means of line segments of the same length and direction, representing equal vectors. The fact that $ABCD$ is a *closed*

Figure 1.17

figure leads immediately to the following *vector sum*:

$$2\boldsymbol{a} + 2\boldsymbol{b} + 2\boldsymbol{c} + 2\boldsymbol{d} = \mathbf{0}.$$

If the vector equation is now multiplied by $\frac{1}{2}$ on both sides, we find with the help of (1.15) that:

$$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d} = \mathbf{0}.$$

Since we may manipulate the vectors in this equation with regard to the $+$ and $-$ operations as we would scalars, we can rewrite this equation as

$$\boldsymbol{d} + \boldsymbol{a} = -\boldsymbol{b} - \boldsymbol{c}.$$

If we inspect Figure 1.17, it follows immediately that $\boldsymbol{e} = \boldsymbol{f}$ and we have show that $EFGH$ is a parallelogram.

V1.6.2. Show that the line that joins the centre points of two sides of a triangle is parallel to the third side, and that the length of this line is half that of the third side.

Figure 1.18

Solution: With the help of the same algebraic properties as those used in the previous problem, it follows in Figure 1.18 that

$$\boldsymbol{d} = \boldsymbol{a} + \boldsymbol{b} = \frac{1}{2}\left(2\boldsymbol{a} + \boldsymbol{b}\right) = \frac{1}{2}\boldsymbol{c}.$$

V1.6.3. Here we derive an *auxiliary result* that enables us to represent the fact that three points lie on a straight line in vector form. In Figure 1.19, A, B, and

Figure 1.19

C are on a straight line so that

$$\overline{BC} = \alpha\,\overline{BA}.$$

The *vector differences* in $\triangle OBC$ and $\triangle OBA$ are substituted in this equation:

$$\boldsymbol{c} - \boldsymbol{b} = \alpha(\boldsymbol{a} - \boldsymbol{b}).$$

If $\boldsymbol{c}$ is now made the subject of the equation (for instance), a very useful result follows:

$$\boldsymbol{c} = \alpha\boldsymbol{a} - (1 - \alpha)\boldsymbol{b}. \tag{1.16}$$

We easily establish that we can also write the equation as

$$\boldsymbol{a} = \beta\boldsymbol{b} - (1 - \beta)\boldsymbol{c}.$$

As in (1.16), the sum of the coefficients on the right is equal to 1.

V1.6.4. Show that the diagonals of a parallelogram bisect each other.

Solution: In Figure 1.20, A, P and C lie on a straight line:

$$\overline{AP} = \alpha\,\overline{AC}.$$

Figure 1.20

Since D, P, and B are also on a straight line, it follows from (1.16) that:

$$\beta \boldsymbol{b} + (1-\beta)\boldsymbol{a} = \alpha(\boldsymbol{b} + \boldsymbol{a}).$$

The algebraic properties of vectors now allow us to collect the terms in $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$(\beta - \alpha)\boldsymbol{b} = (\alpha + \beta - 1)\boldsymbol{a}.$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ do not have the same *direction*, this equation can be true only if the *null vector* occurs on both sides, that is,

$$\beta - \alpha = 0,$$
$$\alpha + \beta = 1.$$

These simultaneous equations can be solved to obtain the values

$$\alpha = \beta = \frac{1}{2}$$

which then proves the result.

Problems

P1.6.1. Show graphically that $-(\boldsymbol{a} - \boldsymbol{b}) = -\boldsymbol{a} + \boldsymbol{b}$.

P1.6.2. A regular hexagon $ABCDEF$ is formed by six directed line segments, all with the same *length*. Let $\overline{FA}$ and $\overline{AB}$ represent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. In terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, find the other sides of the hexagon as well as $\overline{AC}$, $\overline{AD}$ and $\overline{AE}$.

P 1.6.3. Consider any triangle $\triangle OAB$ with a point C on AB. Let $\boldsymbol{a} = \overline{AB}$, $\boldsymbol{b} = \overline{OB}$ and $\boldsymbol{c} = \overline{OC}$. It follows from (1.16) that $\boldsymbol{c} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}$, where $\lambda + \mu = 1$. Find the values of λ and μ if

(a) C is the midpoint of AB.

(b) A is the midpoint of CB.

(c) C is between A and B with $AC = \frac{1}{3}AB$.

P1.6.4. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are given vectors representing the diagonals of a parallelogram, find the sides of the paralellogram.

P1.6.5. In $\triangle ABC$, P, Q and R are the centre points of sides AB, BC and CA respectively. Show that for any point O, $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OP} + \overline{OQ} + \overline{OR}$.

P1.6.6. Show that there exists a triangle with sides which are the medians of any given triangle.

P1.6.7. Show that the medians of a triangle meet in a common point which is a point of trisection of the medians.

P1.6.8. In Figure 1.21, $ABCD$ is a parallelogram with P and Q the midpoints of sides BC and CD respectively. Show that AP and AQ trisect diagonal BD at points E and F.

1.7 Component form of vectors

1.7.1 Unit vector

Consider the vector

$$\boldsymbol{q} = \left(\frac{1}{a}\right)\boldsymbol{a}.$$

It follows from definition that

$$|\boldsymbol{q}| = \left(\frac{1}{a}\right)|\boldsymbol{a}| = 1.$$

Figure 1.21

Vector $\boldsymbol{q}$ is thus a vector with the *same direction* as $\boldsymbol{a}$ and *magnitude 1.* We shall refer to this as a *unit vector* and indicate it by $\hat{a}$. We shall also often use the shorthand method of writing $\dfrac{\boldsymbol{a}}{a}$ instead of $\left(\dfrac{1}{a}\right)\boldsymbol{a}$, that is,

$$\hat{a} = \left(\frac{1}{a}\right)\boldsymbol{a} := \frac{\boldsymbol{a}}{a}. \tag{1.17}$$

It is sometimes useful to rewrite (1.17) as

$$\boldsymbol{a} = a\hat{a}. \tag{1.18}$$

In this form we can read directly the *magnitude* of $\boldsymbol{a}$ on the right, while vector $\hat{a}$ is associated only with the direction of $\hat{a}$. It is very convenient to write vector $\boldsymbol{a}$ *formally* in this way as the "product of magnitude a and direction $\hat{a}$".

It is important to note that $\hat{a}$ is *dimensionless*: if $\boldsymbol{a}$ is a displacement given in metres, the dimensions of both $\boldsymbol{a}$ and a in (1.18) are those of displacement, and $\hat{a}$ must therefore be dimensionless. It is however true that $\hat{a}$ indicates the *scale* in which the vector is represented. If the scale used in Figure 1.8 is 1 cm : 4 Newton, for instance, PQ will represent 16 Newton because it is 4 cm long.

The unit vectors in the *Cartesian coordinate directions* are very important. We shall refer to these as $(\hat{x}, \hat{y}, \hat{z})$ and they are illustrated in Figure 1.22. Other notations that occur in literature are $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ or $(\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z)$, for instance.

Figure 1.22

1.7.2 Components and projections of vectors

We became acquainted with *components* of *displacements* in §1.3.2. It is useful to focus our attention here on triangle $\triangle P_1QP_2$, found in Figure 1.6(b) and reproduced in Figure 1.23. From the latter figure we have the following for the x component of the displacement:

$$a_x = a\cos\alpha$$

Following the above, we define the e component (for instance) of any vector $\boldsymbol{a}$ as

Figure 1.23

the product of $\boldsymbol{a}$'s magnitude with the cosine of the *smallest angle* between $\boldsymbol{a}$ and $\hat{e}$'s *forward* directions, and we denote it as a_e. Therefore, for the vectors in Figure

1.24, we write:

$$a_e := a\cos\theta. \tag{1.19}$$

The *projection* of $\boldsymbol{a}$ in the e direction is a *vector* indicated by $\boldsymbol{a}_e$ and defined as

Figure 1.24

$$\boldsymbol{a}_e = a_e\hat{e}. \tag{1.20}$$

The geometric association between the representations of vectors and the representations of their projections in a given direction is shown for different values of θ in Figure 1.25. Note the association between the sign of a vector's e component

Figure 1.25

and the direction of its e projection: for an acute (obtuse) θ, the component is positive (negative) and the projection is parallel (opposite) to the vector.

1.7.3 Summation of vectors in terms of components

The usefulness of the components of a vector is evident from the following important result:

Theorem 1. The e component of the vector sum of two vectors is equal to the sum of the e components of the vectors.

Figure 1.26

Proof. The following is valid in Figure 1.26:

$$\boldsymbol{c}_e = \boldsymbol{a}_e + \boldsymbol{b}_e.$$

It then follows from definition (1.20) that

$$c_e\hat{e} = a_e\hat{e} + b_e\hat{e}$$

The algebraic properties of the product of a scalar and a vector allow us to collect the terms in $\hat{e}$ on the right-hand side:

$$c_e\hat{e} = (a_e + b_e)\hat{e}.$$

Since the magnitudes of the vectors on both sides of the equation must be equal, the required result follows:

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} \quad \Longrightarrow \quad c_e = a_e + b_e. \tag{1.21}$$

□

1.7.4 Component form

In Figure 1.27, the representation of vector $\boldsymbol{a}$ is shown with a chosen reference system. It is evident that $\boldsymbol{a}$ is equal to the *sum* of its x, y and z *projections*:

$$\boldsymbol{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}. \tag{1.22}$$

We shall refer to (1.22) as the *component form* of a vector. Since the components of $\boldsymbol{a}$ are shown explicitly in this form, it will become obvious that there are huge advantages to this representation of a vector. In fact, we can now reformulate a number of algebraic properties of vectors in terms of the components of vectors.

(a) *Magnitude and Direction*: If Figure 1.27 is compared to Figure 1.6(b), it is evident that the results in equations (1.1b) and (1.1c) can be obtained for any vector. The *magnitude* and *direction* of $\boldsymbol{a}$ can therefore be calculated as in (1.1), that is, the vector in (1.22) has the magnitude

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{1.23a}$$

while the *direction* thereof is contained in the *direction cosines*

$$\begin{aligned} \ell &:= \cos\alpha = \frac{a_x}{a} \\ m &:= \cos\beta = \frac{a_y}{a} \\ n &:= \cos\gamma = \frac{a_z}{a} \end{aligned} \tag{1.23b}$$

(b) *Vector Equality:* Let

$$\boldsymbol{a} = \boldsymbol{b}. \tag{1.24a}$$

We write both vectors in component form:

$$a_x\hat{x} + a_y\hat{y} + a_z\hat{z} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}.$$

Standard manipulation yields

$$(a_x - b_x)\hat{x} + (a_y - b_y)\hat{y} + (a_z - b_z)\hat{z} = \mathbf{0}.$$

It follows from (1.19) that *all* the *components* of the null vector are zero, and hence

$$a_x = b_x, \qquad a_y = b_y, \qquad a_z = b_z. \tag{1.24b}$$

Vector equality implies that the *corresponding components* of vectors are the same.

(c) *Vector Sum*: Let

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}. \tag{1.25a}$$

We write the vectors on the right-hand side in component form and again perform standard manipulation:

$$\begin{aligned} \boldsymbol{c} &= (a_x\hat{x} + a_y\hat{y} + a_z\hat{z}) + (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \\ &= (a_x + b_x)\hat{x} + (a_y + b_y)\hat{y} + (a_z + b_z)\hat{z}. \end{aligned}$$

It follows from (1.24) that

$$c_x = a_x + b_x, \qquad c_y = a_y + b_y, \qquad c_z = a_z + b_z. \tag{1.25b}$$

The *Cartesian components* of vectors add in the same way as the *vectors themselves*; a result that is obviously only a special case of the theorem in §1.7.3.

(d) *Product of a Scalar and a Vector*: In terms of the components of $\boldsymbol{a}$, we have

$$\lambda\boldsymbol{a} = \lambda(a_x\hat{x} + a_y\hat{y} + a_z\hat{z}.$$

From the algebraic properties of the product of a scalar and a vector then follows

$$\lambda\boldsymbol{a} = (\lambda a_x)\hat{x} + (\lambda a_y)\hat{y} + (\lambda a_z)\hat{z} \tag{1.26}$$

Hence, if a vector is multiplied by a scalar λ *each of its components* is multiplied by the factor λ.

(e) *Components of Unit Vectors*: Consider

$$\hat{a} = \left(\frac{1}{a}\right)\boldsymbol{a} = \left(\frac{a_x}{a}\right)\hat{x} + \left(\frac{a_y}{a}\right)\hat{y} + \left(\frac{a_z}{a}\right)\hat{z} \qquad (1.27)$$
$$= \ell\hat{x} + m\hat{y} + n\hat{z}.$$

The components of $\hat{a}$ are the *direction cosines* of $\boldsymbol{a}$.

1.7.5 Position Vector

Figure 1.28

The position vector of P, indicated by $\boldsymbol{r}$, is the vector that links the *origin* to P, as shown in Figure 1.28. It is evident that $\boldsymbol{r}$ is a *displacement.* It follows from (1.1a) for the components of $\boldsymbol{r}$ that:

$$r_x = x - 0, \quad r_y = y - 0, \quad r_z = z - 0.$$

The position vector of P is thus a vector, the *components* of which are the *coordinates* of P, that is,

$$\boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}. \qquad (1.28)$$

Of course we can also write $\boldsymbol{r}$ formally in the form

$$\boldsymbol{r} = r\hat{r}, \qquad (1.29)$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1.30}$$

is the distance between O and P, and where $\hat{r}$ is a unit vector with the same direction as $\boldsymbol{r}$. It is convenient to deal with *displacements* in terms of position vectors. In Figure 1.29, we have the following for the displacement from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$

$$\overline{P_1P_2} = \boldsymbol{r}_2 - \boldsymbol{r}_1,$$

where

$$\boldsymbol{r}_1 = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}, \quad \boldsymbol{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z},$$

so that

$$\overline{P_1P_2} = (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z}. \tag{1.31}$$

Equation (1.31) is of course only an alternative way of writing (1.1a).

Figure 1.29

Examples

V1.7.1. Find the magnitude of the displacement from $P_1(-1, 3, 4)$ to $P_2(2, -3, 2)$ as well as the angles that the directed line segment $\overline{P_1P_2}$ forms with the coordinate directions.

Solution: In Figure 1.29, the following applies to this special case

$$\boldsymbol{r}_1 = -\hat{x} + 3\hat{y} + 4\hat{z}; \quad \boldsymbol{r}_2 = 2\hat{x} - 3\hat{y} + 2\hat{z}.$$

The displacement is then given in component form by

$$\begin{aligned} \boldsymbol{a} &:= \overline{P_1P_2} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = \{2-(-1)\}\hat{x} + \{-3-3\}\hat{y} + \{2-4\}\hat{z} \\ &= 3\hat{x} - 6\hat{y} - 2\hat{z}. \end{aligned}$$

The magnitude of the displacement is obtained by means of (1.23a):

$$a = \sqrt{3^2 + (-6)^2 + (-2)^2} = 7$$

The direction cosines of $\boldsymbol{a}$ are obtained from (1.23b):

$$\ell = \frac{3}{7}; \quad m = \frac{-6}{7}; \quad n = \frac{-2}{7}.$$

The angles that $\boldsymbol{a}$ forms with the positive X, Y and Z directions then are

$$\alpha = 64.62°; \quad \beta = 149.00°; \quad \gamma = 106.60°.$$

V1.7.2. Find, in component form, a unit vector with the same direction as the position vector of point (2,1,-2).

Solution: We know that the components of the position vector of a point is given by the coordinates of the point. Therefore, in this case we have

$$\boldsymbol{r} = 2\hat{x} + \hat{y} - 2\hat{z}.$$

The magnitude of this vector follows immediately from (1.30):

$$r = \sqrt{2^2 + 1^2 + (-2)^2} = 3.$$

It follows from definition that

$$\begin{aligned} \hat{r} = \frac{\boldsymbol{r}}{r} &= \frac{2\hat{x} + \hat{y} - 2\hat{z}}{3} \\ &= \left(\frac{2}{3}\right)\hat{x} + \left(\frac{1}{3}\right)\hat{y} - \left(\frac{2}{3}\right)\hat{z}. \end{aligned}$$

Problems

P1.7.1. Which requirements must $\hat{a}$ and $\hat{b}$ meet so that $\hat{a}+\hat{b}$ is also a unit vector?

P1.7.2. Solve the equation $\boldsymbol{r} = a_1\hat{x}+a_2\hat{y}$ for a_1 and a_2 if $\boldsymbol{r}$ is a vector of magnitude 10, which is found in the first quadrant of the XY plane and forms an angle of $30°$ with the positive X direction.

P1.7.3. Find the magnitudes of the following vectors as well as the angles that they form with the coordinate directions:

(a) $-14\hat{x} + 7\hat{y} + 14\hat{z}$

(b) $-3\hat{x} + 6\hat{y} - 2\hat{z}$

(c) $8\hat{x} - 8\hat{y} + 14\hat{z}$

P1.7.4. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are the position vectors of points $(3, -1, 4)$, $(-2, 4, -3)$ and $(1, 2, -1)$ respectively. Find:

(a) $2\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}$ in component form,

(b) $|\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}|$,

(c) $|3\boldsymbol{a} - 2\boldsymbol{b} + 4\boldsymbol{c}|$ and

(d) a unit vector with the same direction as $3\boldsymbol{a} - 2\boldsymbol{b} + 4\boldsymbol{c}$.

P1.7.5. A particle is displaced from point $(1, 2, 3)$ over a distance of 6 metres in a direction, the direction cosines of which are given as -0.81, 0.32 and -0.49. What are the coordinates of the end point of the displacement?

P1.7.6. Let $\boldsymbol{a} = \hat{x} - 2\hat{y} + 3\hat{z}$, $\boldsymbol{b} = 3\hat{x} + 4\hat{y} - \hat{z}$ and $\boldsymbol{c} = -2\hat{x} - 6\hat{y} + 4\hat{z}$.

(a) Show that these vectors can form the sides of a triangle.

(b) Calculate the lengths of the medians of the triangle.

P 1.7.7. Find the equations of the straight line between points $(2, 3, -1)$ and $(4, -2, 0)$.

1.8 Scalar product

1.8.1 Definition

The *scalar product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is indicated by $\boldsymbol{a} \cdot \boldsymbol{b}$ and defined as the product of their magnitudes and the cosine of the smallest angle between their forward directions. Therefore, for the two vectors respresented in Figure 1.30, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} := ab \cos\theta \tag{1.32}$$

In the literature, this product is often referred to as the inner product, the dot

Figure 1.30

product or the point product. All three names may occur in the rest of these notes.

1.8.2 Special cases

(a) For the scalar product of a vector with itself, we have

$$\boldsymbol{a} \cdot \boldsymbol{a} = (a)(a)\cos 0^\circ = a^2.$$

This product is also often indicated in the literature as $\boldsymbol{a}^2$ ($\overline{a}^2$ in written work) so that

$$\boldsymbol{a}^2 := \boldsymbol{a} \cdot \boldsymbol{a} = a^2. \tag{1.33}$$

(b) It follows from definition (1.32) that the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ is equal to zero if

$$\boldsymbol{a} = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{b} = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{a} \perp \boldsymbol{b}. \tag{1.34}$$

(c) If $\hat{e}$ and $\hat{f}$ are two *unit vectors*, and θ is the angle between their forward directions, it follows that

$$\hat{e} \cdot \hat{f} = (1)(1)\cos\theta = \cos\theta. \tag{1.35}$$

(d) It follows for the *Cartesian unit vectors* that for instance

$$\hat{x} \cdot \hat{x} = (1)(1)\cos 0^\circ = 1$$

and

$$\hat{x} \cdot \hat{y} = (1)(1)\cos 90^\circ = 0.$$

Now we can easily write down all the possible scalar products (up to interchanges within the scalar product) between the Cartesian unit vectors:

$$\begin{aligned} \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \\ \hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0 \end{aligned} \tag{1.36}$$

(e) For vectors $\boldsymbol{a}$ and $\hat{e}$ in Figure 1.24, we have

$$\boldsymbol{a} \cdot \hat{e} = (a)(1)\cos\theta.$$

It then follows from definition (1.19) that

$$\boldsymbol{a} \cdot \hat{e} = a_e. \tag{1.37}$$

1.8.3 Algebraic Properties of the Scalar Product

(a) The scalar product is *not closed*—in definition (1.32) we see that two vectors are mapped to a scalar by means of the scalar product.

(b) The scalar product is *commutative*—since scalars are commutative with respect to multiplication, it follows for the vectors in Figure 1.30 that

$$\boldsymbol{b} \cdot \boldsymbol{a} = ba \cos\theta = ab \cos\theta,$$

so that

$$\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{1.38}$$

(c) The scalar product is *associative* with regard to the product of a scalar and a vector—since scalars are associative and commutative with regard to multiplication, it follows for the vectors in Figure 1.31 that

$$(\alpha \boldsymbol{a}) \cdot (\beta \boldsymbol{b}) = (\alpha a)(\beta b) \cos\theta = (\alpha\beta)(ab \cos\theta).$$

We see therefore, that scalars that occur in a scalar product can be manip-

Figure 1.31

ulated as if they occur in a product of scalars:

$$(\alpha \boldsymbol{a}) \cdot (\beta \boldsymbol{b}) = (\alpha\beta) \boldsymbol{a} \cdot \boldsymbol{b} \tag{1.39}$$

(d) The scalar product is *distributive*—from (1.21), it follows in Figure 1.32 that

$$d_c = a_c + b_c.$$

It then follows from (1.37) that

Figure 1.32

$$\boldsymbol{d} \cdot \hat{c} = \boldsymbol{a} \cdot \hat{c} + \boldsymbol{b} \cdot \hat{c}.$$

If we multiply this equation by c, we can use property (c) above to write the equation in the following form:

$$\boldsymbol{d} \cdot (c\hat{c}) = \boldsymbol{a} \cdot (c\hat{c}) + \boldsymbol{b} \cdot (c\hat{c})$$

Since $c\hat{c} = \boldsymbol{c}$, the distributivity of the scalar product follows immediately:

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c} \tag{1.40}$$

1.8.4 Scalar product in terms of the components of vectors

We now write $\boldsymbol{a} \cdot \boldsymbol{b}$ in terms of the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ by substituting the component form of both vectors into the scalar product, and by using properties

(1.38) to (1.40) as well as scalar products (1.36):

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= (a_x\hat{x}+a_y\hat{y}+a_z\hat{z})\cdot(b_x\hat{x}+b_y\hat{y}+b_z\hat{z})\\
&= (a_x\hat{x})\cdot(b_x\hat{x})+(a_x\hat{x})\cdot(b_y\hat{y})+(a_x\hat{x})\cdot(b_z\hat{z})\\
&\quad+(a_y\hat{y})\cdot(b_x\hat{x})+(a_y\hat{y})\cdot(b_y\hat{y})+(a_y\hat{y})\cdot(b_z\hat{z})\\
&\quad+(a_z\hat{z})\cdot(b_x\hat{x})+(a_z\hat{z})\cdot(b_y\hat{y})+(a_z\hat{z})\cdot(b_z\hat{z})\\
&= (a_xb_x)\hat{x}\cdot\hat{x}+(a_xb_y)\hat{x}\cdot\hat{y}+(a_xb_z)\hat{x}\cdot\hat{z}\\
&\quad+(a_yb_x)\hat{y}\cdot\hat{x}+(a_yb_y)\hat{y}\cdot\hat{y}+(a_yb_z)\hat{y}\cdot\hat{z}\\
&\quad+(a_zb_x)\hat{z}\cdot\hat{x}+(a_zb_y)\hat{z}\cdot\hat{y}+(a_zb_z)\hat{z}\cdot\hat{z}\\
&= a_xb_x(1)+a_xb_y(0)+a_xb_z(0)\\
&\quad+a_yb_x(0)+a_yb_y(1)+a_yb_z(0)\\
&\quad+a_zb_x(0)+a_zb_y(0)+a_zb_z(1)
\end{aligned}$$

Here we obtain a *very important result*:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_xb_x+a_yb_y+a_zb_z \tag{1.41}$$

Together with definition (1.32), this result enables us to determine the *angles* between the lines in the space and to solve a large number of geometric problems in an *algebraic* way.

Examples

V1.8.1. If $\boldsymbol{a} = 3\hat{x}-5\hat{y}+3\hat{z}$ and $\boldsymbol{b} = -2\hat{x}+\hat{y}+2\hat{z}$, find $\boldsymbol{a}\cdot\boldsymbol{b}$.

Solution: From (1.41) we have

$$\begin{aligned}
\boldsymbol{a}\cdot\boldsymbol{b} &= (3)(-2)+(-5)(1)+(3)(2)\\
&= -6-5+6\\
&= -5
\end{aligned}$$

V1.8.2. Determine the component of $\boldsymbol{a} = 9\hat{x}-3\hat{y}+6\hat{z}$ in the direction of vector $\boldsymbol{e} = 4\hat{x}+2\hat{y}-4\hat{z}$.

Solution: We first determine a unit vector with the same direction as $\boldsymbol{e}$:

$$e = \sqrt{(4)^2 + (2)^2 + (-4)^2} = 6$$

and hence

$$\hat{e} = \frac{\boldsymbol{e}}{e} = \left(\frac{2}{3}\right)\hat{x} + \left(\frac{1}{3}\right)\hat{y} - \left(\frac{2}{3}\right)\hat{z}.$$

From (1.37) it then follows that:

$$\begin{aligned} a_e = \boldsymbol{a} \cdot \hat{e} &= (9)\left(\frac{2}{3}\right) + (-3)\left(\frac{1}{3}\right) + (6)\left(-\frac{2}{3}\right) \\ &= 6 - 1 - 4 \\ &= 1. \end{aligned}$$

V 1.8.3. Find the angle which the line between the points $(2, 1, 2)$ and $(3, 4, 0)$ spans at the origin.

Solution: Firstly, we write $\boldsymbol{a}$ and $\boldsymbol{b}$ in Figure 1.33 in component form:

$$\boldsymbol{a} = 2\hat{x} + \hat{y} + 2\hat{z}, \quad \boldsymbol{b} = 3\hat{x} + 4\hat{y}.$$

By combining (1.32) and (1.41), we then have

Figure 1.33

$$\begin{aligned}\cos\theta &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{ab}\\ &= \frac{(2)(3)+(1)(4)+(2)(0)}{(3)(5)}\\ &= \frac{2}{3},\end{aligned}$$

and hence

$$\theta = 48.19°.$$

Problems

P1.8.1. Find the scalar product between the following vectors:

(a) $3\hat{x} - 5\hat{y} - 2\hat{z}$ and $-6\hat{x} + 7\hat{y} - \hat{z}$,

(b) $\hat{x} - 2\hat{y} + 3\hat{z}$ and $\hat{y} - \hat{z}$,

(c) $2\hat{x} + \hat{y}$ and $2\hat{x} - \hat{y}$.

P1.8.2. Find the angles between:

(a) $2\hat{x} - 3\hat{y} + \hat{z}$ and $2\hat{x} - 6\hat{y}$,

(b) the position vectors of the points with coordinates $(3, 0, 5)$ and $(3, -2, -2)$,

(c) lines AB and AC where A, B and C are points with coordinates $(2, -1, 3)$, $(3, -2, 7)$ and $(1, -1, 6)$ respectively, and

(d) two line segments with direction cosines $\left\{\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\}$ and $\left\{\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}$.

P1.8.3. Find the angle between two diagonals of a cube.

P1.8.4. Write $2a + 3b + 4c$ as the scalar product of two vectors.

P1.8.5. Show that $\left(\frac{\boldsymbol{a}}{a^2} - \frac{\boldsymbol{b}}{b^2}\right)^2 = \frac{(\boldsymbol{a}-\boldsymbol{b})^2}{a^2b^2}$.

P1.8.6. Find the components of a vector with magnitude 3 which forms equal angles with $-\hat{x}$, $-\hat{y}$ en $\hat{z}$.

P1.8.7. Show that vectors $\boldsymbol{a} = 3\hat{x} - 2\hat{y} + \hat{z}$, $\boldsymbol{b} = \hat{x} - 3\hat{y} + 5\hat{z}$ and $\boldsymbol{c} = 2\hat{x} + \hat{y} - 4\hat{z}$ form the side of a right-angled triangle.

P1.8.8. Find, in component form, the projections of $\boldsymbol{a} = \hat{x} - 2\hat{y} + \hat{z}$ respectively parallel and perpendicular to $\boldsymbol{b} = 4\hat{x} - 4\hat{y} + 7\hat{z}$.

P1.8.9. The vertices of a triangle have $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as position vectors. Show that angle θ at the vertex with position vector $\boldsymbol{c}$ is given by

$$\cos\theta = \frac{(\boldsymbol{a}-\boldsymbol{c})\cdot(\boldsymbol{b}-\boldsymbol{c})}{\left|\big((\boldsymbol{a}-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{c})\big)\big((\boldsymbol{b}-\boldsymbol{c})\cdot(\boldsymbol{b}-\boldsymbol{c})\big)\right|^{\frac{1}{2}}}.$$

P1.8.10. It is given that $\hat{e}$, $\hat{f}$ and $\hat{e}-\hat{f}$ are all unit vectors. Use the scalar product to obtain further information about $\hat{e}$ and $\hat{f}$.

P1.8.11. Vector $\hat{e} - 2\hat{f}$ has a magnitude of 2; $\hat{e}$ and $\hat{f}$ are unit vectors. Find the angle between $\hat{e}$ and $\hat{f}$.

P1.8.12. Under which conditions is $\boldsymbol{a}\cdot\boldsymbol{c} = \boldsymbol{b}\cdot\boldsymbol{c}$ true?

P1.8.13. If $\boldsymbol{a} = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{b}$ what could be said about $\boldsymbol{b}$?

P1.8.14. Find the projection of $\boldsymbol{r} = 5\hat{x} + 2\hat{y} - 3\hat{z}$ on the XY plane.

P1.8.15. The vector $a\hat{x} - 2\hat{y} + \hat{z}$ is perpendicular to vector $\hat{x} - 2\hat{y} - 3\hat{z}$. Find a.

P1.8.16. Vectors $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ are perpendicular to each other. Use the scalar product to reach a conclusion about the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$.

P1.8.17. Find the projection of $6\hat{x} - 6\hat{y} - 7\hat{z}$ on

(a) the Y axis, and

(b) $3\hat{y} - 4\hat{z}$.

P1.8.18. Unit vector $\hat{e}$ forms angles of 45°, 60° and 120° with the X, Y and Z axes respectively. Find the e component of vector $\hat{x} - \hat{y} - \hat{z}$.

P1.8.19. Use the scalar product to prove the cosine rule.

P1.8.20. Prove that an angle inscribed in a semi-circle is a right angle.

P1.8.21. $ABCD$ is a parallelogram. Show that $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2$.

P1.8.22. Prove the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ by using the scalar product between two unit vectors in the XY plane.

P1.8.23. Find the shortest distance between point $(4, 6, -4)$ and the line between points $(2, 2, 1)$ and $(4, 3, -1)$.

1.9 Vector product

1.9.1 Definition

The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is indicated by $\boldsymbol{a} \times \boldsymbol{b}$ and it is a *vector*, which has

- its magnitude as the product of the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ and the sine of the smallest angle between their forward directions, and
- a direction that is established as follows: if the spiral part of a corkscrew is placed *perpendicular* to *both* $\boldsymbol{a}$ and $\boldsymbol{b}$ and the handle of the corkscrew is turned along the *shortest route from* $\boldsymbol{a}$ *to* $\boldsymbol{b}$, the spiral part moves in the direction of $\boldsymbol{a} \times \boldsymbol{b}$.

In Figure 1.34, where $\boldsymbol{a}$ and $\boldsymbol{b}$ both lie on the plane of the page, and $\hat{n}$ is directed perpendiculary *out of* the page, we therefore have

$$\boldsymbol{a} \times \boldsymbol{b} := (ab\sin\theta)\hat{n} \tag{1.42}$$

It is very important to note the role that the *order* of the vectors plays in the vector product: when establishing the direction of the product, the handle of the corkscrew is rotated *from* the vector that occurs *first* in the product *to* the vector that occurs *last*.

Figure 1.34

1.9.2 Special cases

(a) For two parallel or antiparallel vectors, as in Figure 1.35, the following applies

$$|\boldsymbol{a} \times \boldsymbol{b}| = ab \sin 0^\circ = 0 \quad \text{or} \quad |\boldsymbol{a} \times \boldsymbol{b}| = ab \sin 180^\circ = 0,$$

and hence

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}. \tag{1.43}$$

A special case of this is

Figure 1.35

$$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}. \tag{1.44}$$

(b) For the Cartesian unit vectors, shown in Figure 1.36, we know from (1.44) that any vector product of two identical unit vectors is equal to the null vector. For two different vectors we for instance have

$$\hat{x} \times \hat{y} = (1)(1)\sin 90^\circ \hat{z}.$$

Figure 1.36

We can easily establish all the possible mutual vector products (up to interchanges within the product) between the three unit vectors:

$$\begin{array}{lll} \hat{x} \times \hat{x} = \mathbf{0} & \hat{y} \times \hat{y} = \mathbf{0} & \hat{z} \times \hat{z} = \mathbf{0} \\ \hat{x} \times \hat{y} = \hat{z} & \hat{y} \times \hat{z} = \hat{x} & \hat{z} \times \hat{x} = \hat{y} \end{array} \tag{1.45}$$

1.9.3 Algebraic Properties of the Vector Product

(a) The vector product is *closed*: The vector product of two vectors is by definition also a *vector*.

(b) The vector product is *anticommutative*: From definition (1.42) follows for the vector in Figure 1.34 that

$$\boldsymbol{b} \times \boldsymbol{a} = ba \sin\theta(-\hat{n}).$$

Therefore, we have the extraordinary property of the vector product that the interchange of the two vectors which occurs in it, changes the sign of the product:

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}. \tag{1.46}$$

It is obvious that we shall have to take note of the *order* of the vectors that occur in the vector product. If we had used the reverse order in the fourth relation in (1.45), we would have had $\hat{y} \times \hat{x} = -\hat{z}$. (Verify that this result is also obtained directly from definition (1.42).)

(c) The vector product is *associative* with regard to multiplication with scalars: In Figure 1.37, we have

$$(\alpha \boldsymbol{a}) \times (\beta \boldsymbol{b}) = (\alpha a)(\beta b) \sin\theta \hat{n} = (\alpha\beta) ab \sin\theta \hat{n},$$

where we used the properties of the product of a scalar and a vector in the

Figure 1.37

last step. The associativity, as described above, follows immediately:

$$(\alpha \boldsymbol{a}) \times (\beta \boldsymbol{b}) = (\alpha\beta) \boldsymbol{a} \times \boldsymbol{b} \tag{1.47}$$

(d) The vector products with $\boldsymbol{a}$ (for instance) of vectors of which projections *perpendicular* to $\boldsymbol{a}$ are identical, are equal. In Figure 1.38, $\boldsymbol{a}$ and $\boldsymbol{b}$ lie in the plane of the page, $\hat{n}$ points perpendicularly out of the page, and $\boldsymbol{b}_\perp$ is the projection of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$. It then follows that

$$\boldsymbol{a} \times \boldsymbol{b} = a(b \sin\theta)\hat{n} = a\,|\boldsymbol{b}_\perp|\,\hat{n} = a\,|\boldsymbol{b}_\perp| \sin 90^\circ \hat{n},$$

and hence

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}_\perp. \tag{1.48}$$

For vector $\boldsymbol{c}$, it then follows that

Figure 1.38

$$\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c}_{\perp} = \boldsymbol{a} \times \boldsymbol{b}_{\perp} = \boldsymbol{a} \times \boldsymbol{b}.$$

(e) The vector product is *distributive*: We now want to show that

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}. \tag{1.49}$$

Although this properties of the the vector product looks just like that of the product of scalars, in view of anticommutativity of the vector product, it is very important to note the *orders* in 1.49: the pairs $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{a}, \boldsymbol{c}\}$ occur in the same order on *both* sides of the equation.

(i) First, we show that (1.49) applies in the special case where $\boldsymbol{a}$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$. As shown in Figure 1.39, $\boldsymbol{b}$ and $\boldsymbol{c}$ both lie in the S plane, which is *perpendicular* to $\boldsymbol{a}$. As in Figure 1.12, $\boldsymbol{b} + \boldsymbol{c}$ can be represented by the diagonal of a parallelogram, which lies in S, the sides of which represent $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

The *magnitudes* of the products that occur in (1.49) can now be written as follows:

$$
\begin{aligned}
|\boldsymbol{a} \times \boldsymbol{b}| &= ab \sin 90^\circ = a\,|\boldsymbol{b}| \\
|\boldsymbol{a} \times \boldsymbol{c}| &= ac \sin 90^\circ = a\,|\boldsymbol{c}| \\
|\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})| &= a\,|\boldsymbol{b} + \boldsymbol{c}| \sin 90^\circ = a\,|\boldsymbol{b} + \boldsymbol{c}|
\end{aligned}
$$

Since all the products in (1.49) contain $\boldsymbol{a}$ and are therefore perpendicular to $\boldsymbol{a}$, all the vectors that are obtained from the products also lie in S. Therefore, vectors $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{b} + \boldsymbol{c}$ are in effect rotated through 90° in S and increased in magnitude by a factor a. This means that $\boldsymbol{a} \times (\boldsymbol{b}+\boldsymbol{c})$ is the diagonal of a parallelogram with sides $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{c}$. The vector sum in (1.49) follows immediately.

(ii) Now we show that (1.49) also applies in general. As shown in Figure 1.40, we can regard $\boldsymbol{b}$ and $\boldsymbol{c}$ both as the sum of their respective projections perpendicular to $\boldsymbol{a}$ (indicated by the subscript $_\perp$) and parallel to $\boldsymbol{a}$ (indicated by the subscript $_\parallel$), that is,

$$
\boldsymbol{b} = \boldsymbol{b}_\perp + \boldsymbol{b}_\parallel, \qquad \boldsymbol{c} = \boldsymbol{c}_\perp + \boldsymbol{c}_\parallel.
$$

Consider

$$
\begin{aligned}
\boldsymbol{d} &:= \boldsymbol{b} + \boldsymbol{c} \\
&= (\boldsymbol{b}_\perp + \boldsymbol{c}_\perp) + (\boldsymbol{b}_\parallel + \boldsymbol{c}_\parallel) \\
&= \boldsymbol{d}_\perp + \boldsymbol{d}_\parallel,
\end{aligned}
$$

where $\boldsymbol{d}_\perp = \boldsymbol{b}_\perp + \boldsymbol{c}_\perp$ and $\boldsymbol{d}_\parallel = \boldsymbol{b}_\parallel + \boldsymbol{c}_\parallel$ are the respective projections of $\boldsymbol{d}$ perpendicular and parallel to $\boldsymbol{a}$. It follows from (1.48) that

$$
\begin{aligned}
\boldsymbol{a} \times \boldsymbol{d} &= \boldsymbol{a} \times \boldsymbol{d}_\perp \\
&= \boldsymbol{a} \times (\boldsymbol{b}_\perp + \boldsymbol{c}_\perp).
\end{aligned}
$$

In the last expression we see the sum of two vectors that are both perpendicular to $\boldsymbol{a}$. In (i) we saw that the distributivity of the vector

Figure 1.40

product is valid in such a case, and with the help of (1.48) we show that (1.49) applies in general:

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \times \boldsymbol{d} \\ &= \boldsymbol{a} \times \boldsymbol{b}_{\perp} + \boldsymbol{a} \times \boldsymbol{c}_{\perp} \\ &= \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}. \end{aligned}$$

1.9.4 Component form of the Vector Product

We now write $\boldsymbol{a} \times \boldsymbol{b}$ in terms of the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ by introducing the component form of both vectors into the vector product and then using characteristics

(1.46) as well as the vector products (1.45):

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= (a_x\hat{x} + a_y\hat{y} + a_z\hat{z}) \times (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \\
&= (a_x\hat{x}) \times (b_x\hat{x}) + (a_x\hat{x}) \times (b_y\hat{y}) + (a_x\hat{x}) \times (b_z\hat{z}) \\
&\quad + (a_y\hat{y}) \times (b_x\hat{x}) + (a_y\hat{y}) \times (b_y\hat{y}) + (a_y\hat{y}) \times (b_z\hat{z}) \\
&\quad + (a_z\hat{z}) \times (b_x\hat{x}) + (a_z\hat{z}) \times (b_y\hat{y}) + (a_z\hat{z}) \times (b_z\hat{z}) \\
&= a_xb_x(\hat{x} \times \hat{x}) + a_xb_y(\hat{x} \times \hat{y}) + a_xb_z(\hat{x} \times \hat{z}) \\
&\quad + a_yb_x(\hat{y} \times \hat{x}) + a_yb_y(\hat{y} \times \hat{y}) + a_yb_z(\hat{y} \times \hat{z}) \\
&\quad + a_zb_x(\hat{z} \times \hat{x}) + a_zb_y(\hat{z} \times \hat{y}) + a_zb_z(\hat{z} \times \hat{z}) \\
&= a_xb_x\mathbf{0} + a_xb_y\hat{z} + a_xb_z(-\hat{y}) \\
&\quad + a_yb_x(-\hat{z}) + a_yb_y\mathbf{0} + a_yb_z\hat{x} \\
&\quad + a_zb_x\hat{y} + a_zb_y(-\hat{x}) + a_zb_z\mathbf{0}
\end{aligned}$$

We then collect the terms in $\hat{x}$, $\hat{y}$ and $\hat{z}$ respectively in order to write the vector product in component form:

$$\boldsymbol{a} \times \boldsymbol{b} = (a_yb_z - a_zb_y)\hat{x} + (a_zb_x - a_xb_z)\hat{y} + (a_xb_y - a_yb_x)\hat{z}. \qquad (1.50)$$

Remark. A convenient way of remembering this result is by writing it in the form of a 3×3 *determinant*:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \qquad (1.51)$$

The determinant is a concept from linear algebra. A 3×3 *matrix* C is a mathematical entity with nine elements ordered as follows:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Note that c_{ij} is the element in the i-th row and j-th column. The *determinant* of this matrix is then denoted and defined as follows:

$$\det C = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$
$$= c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32}$$
$$- c_{11}c_{23}c_{32} - c_{13}c_{22}c_{31} - c_{12}c_{21}c_{33}$$

The 3×3 matrix is of course only a special case of a $n \times n$ matrix.

As in the case of the scalar product, (1.46) and (1.51) can be used together to solve geometric problems, especially in cases where the sine relations of angles are used.

Examples

V1.9.1. Find, in component form, a unit vector perpendicular to both $\boldsymbol{a} = \hat{x} - \hat{y} + 3\hat{z}$ and $\boldsymbol{b} = -\hat{x} - \hat{z}$.

Solution: We know that $\boldsymbol{c} \coloneqq \boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and that $\hat{c} = \boldsymbol{c}/c$ is a unit vector with the same direction as $\boldsymbol{c}$. We first calculate $\boldsymbol{c}$ by the use of (1.51):

$$\begin{aligned}
\boldsymbol{c} &= \boldsymbol{a} \times \boldsymbol{b} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -1 & 3 \\ -1 & 0 & -1 \end{vmatrix} \\
&= [(-1)(-1) - (0)(3)]\hat{x} + [(3)(-1) - (1)(-1)]\hat{y} + [(1)(0) - (-1)(-1)]\hat{z} \\
&= \hat{x} - 2\hat{y} - \hat{z}
\end{aligned}$$

The magnitude of $\boldsymbol{c}$ is obtained from (1.23a)

$$c = \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{6},$$

and hence

$$\hat{c} = \frac{1}{\sqrt{6}}(\hat{x} - 2\hat{y} - \hat{z}).$$

V1.9.2. Calculate the areas of a triangle and a parallelogram with the help of the vector product. *Solution:* In Figure 1.41, it follows for the area of the parallelogram

Figure 1.41

that

$$\begin{aligned} S_p &= ah \\ &= ab \sin\theta \\ &= |\boldsymbol{a} \times \boldsymbol{b}|, \end{aligned}$$

and for that of the triangle

$$\begin{aligned} S_t &= \frac{1}{2}ck \\ &= \frac{1}{2}cd\sin\phi \\ &= \frac{1}{2}|\boldsymbol{c} \times \mathbf{d}|. \end{aligned}$$

V1.9.3. Find the shortest distance from $(6, -4, 4)$ to the line between $(2, 1, 2)$ and $(3, -1, 4)$.

Solution: In Figure 1.42, we have for the shortest distance between R and PQ,

$$h = b\sin\theta = \frac{1}{a}(ab\sin\theta) = \frac{1}{a}|\boldsymbol{a} \times \boldsymbol{b}|.$$

Figure 1.42

We calculate this quantity with the help of (1.1) and (1.51):

$$\boldsymbol{a} = \overline{PQ} = \hat{x} - 2\hat{y} + 2\hat{z}, \qquad \boldsymbol{b} = \overline{PR} = 4\hat{x} - 5\hat{y} + 2\hat{z}$$

so that

$$a = \sqrt{1^2 + (-2)^2 + 2^2} = 3 \text{ m}$$

and

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -2 & 2 \\ 4 & -5 & 2 \end{vmatrix} \\ &= 6\hat{x} + 6\hat{y} + 3\hat{z}. \end{aligned}$$

The required distance then is

$$\begin{aligned} h &= \frac{1}{3}\left|6\hat{x} + 6\hat{y} + 3\hat{z}\right| \\ &= \frac{1}{3}\sqrt{6^2 + 6^2 + 3^2} \\ &= 3 \text{ m}. \end{aligned}$$

Problems

P1.9.1. If $\boldsymbol{a} = 2\hat{x} - 3\hat{y} - \hat{z}$ and $\boldsymbol{b} = \hat{x} + 4\hat{y} - 2\hat{z}$, calculate

(i) $\boldsymbol{a} \times \boldsymbol{b}$,

(ii) $\boldsymbol{b} \times \boldsymbol{a}$, and

(iii) $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})$.

P1.9.2. If $\boldsymbol{a} = 3\hat{x} - \hat{y} + 2\hat{z}$, $\boldsymbol{b} = 2\hat{x} + \hat{y} - \hat{z}$ and $\boldsymbol{c} = \hat{x} - 2\hat{y} + 2\hat{z}$, calculate

(i) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$, and

(ii) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

P1.9.3. Use definitions to prove that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \cdot \boldsymbol{b})$.

P1.9.4. Find a vector parallel to the XY plane and perpendicular to vector $4\hat{x} - 3\hat{y} + \hat{z}$.

P1.9.5. Find the area of a triangle with vertexes $(1, 3, 2)$, $(2, -1, 1)$ and $(-1, 2, 3)$.

P1.9.6. Prove the sine rule for triangles by using the vector product.

P1.9.7. Simplify the following two expressions:

(i) $(\boldsymbol{a} \cdot \boldsymbol{b})^2 + (\boldsymbol{a} \times \boldsymbol{b})^2$, and

(ii) $(\boldsymbol{a} \cdot \boldsymbol{b})^2 - (\boldsymbol{a} \times \boldsymbol{b})^2$.

P 1.9.8. Vector $\boldsymbol{r}$ lies in the XY plane and satisfies the equation $\hat{x} \times \boldsymbol{r} = \hat{z}$. Describe the locus of the points for which $\boldsymbol{r}$ is a position vector.

P1.9.9. Prove the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ by using the vector product of two unit vectors in the XY plane.

P1.9.10. Simplify $(\boldsymbol{a} + \boldsymbol{b}) \cdot [(\boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{c} + \boldsymbol{a})]$.

1.10 Multiple products

Since $\boldsymbol{b} \times \boldsymbol{c}$ is a ***vector***, it can occur in a scalar product or in a vector product, so that multiple products are formed. We shall now discuss two types of multiple product, viz. products of the form $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ from (a scalar triple product) and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ (a vector triple product).

1.10.1 The Scalar Triple Product

Because $(\boldsymbol{a} \cdot \boldsymbol{b}) \times \boldsymbol{c}$ has no significance, $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ can be written *unambiguously* as $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$.

A *geometric significance* can be given to the scalar triple product. Figure 1.43 is a three-dimensional figure, the sides of which are represented as vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Each pair of opposite planes of the figure are in other words identical parallelograms spanned by two of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. This figure is referred to as a *parallelepiped*. We know from geometric considerations that the volume

Figure 1.43

of the parallelepiped is given by the product of the area S of the base with the perpendicular height h. We also saw in V1.9.2 that we can write S as $|\boldsymbol{b} \times \boldsymbol{c}|$.

Therefore, the volume of the figure is

$$\begin{aligned} V &= Sh \\ &= |\boldsymbol{b} \times \boldsymbol{c}| \, (a \cos\theta) \\ &= |\boldsymbol{b} \times \boldsymbol{c}| \, (\boldsymbol{a} \cdot \hat{n}) \\ &= (|\boldsymbol{b} \times \boldsymbol{c}| \, \hat{n}) \cdot \boldsymbol{a}. \end{aligned}$$

Since $\boldsymbol{b} \times \boldsymbol{c}$ is parallel or opposed to $\hat{n}$ (the latter n the case where $\boldsymbol{b}$ and $\boldsymbol{c}$ are interchanged in Figure 1.43), we have

$$V = \pm \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c},$$

or

$$V = |\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}| \, . \tag{1.52}$$

But it follows from Figure 1.43 that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ may occur in any order in (1.52). Therefore, two scalar triple products in which three vectors occur will differ *at the most by a sign*. We can establish the mutual relation between the six possible scalar triple products in which $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ occur by first applying (1.51) in $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$,

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = (a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix},$$

and then (1.41):

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.53}$$

The mutual relations between the scalar triple products that contain $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ follows from a well-known *symmetry property* of the determinant, one that is easily obtained from the definition of the determinant: if the elements of any two rows in the determinant are interchanged, the *sign* of the determinant changes. For the triple product in (1.53), this implies that *its sign changes if any two vectors in the* *product are interchanged.* The following follows from this:

$$\begin{aligned} &\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}=\boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a} \\ &=-\boldsymbol{a}\cdot\boldsymbol{c}\times\boldsymbol{b}=-\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c}=-\boldsymbol{c}\cdot\boldsymbol{b}\times\boldsymbol{a} \end{aligned} \tag{1.54}$$

We see that another way of describing the above symmetry property of the scalar triple product is that $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}$ is *invariant* with regard to the interchange of the $\times$ and the $\cdot$ operations.

The geometric interpretation attached to the scalar triple product enables us to reach a very important conclusion: the scalar triple product $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}$ is equal to zero if:

(1) $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar or

(2) any two of these three vectors are multiples of each other or

(3) any two of these three vectors are equal.

Remark. Naturally, cases (2) and (3) are only special cases of (1), but it is still good to point them out.

The latter properties of the scalar triple product as well as prescription (1.53) for the calculation thereof enable us to solve geometric problems involving planes in an algebraic way.

Examples

V 1.10.1. Calculate $\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}$ where $\boldsymbol{a}=\hat{x}+\hat{y}+\hat{z}$, $\boldsymbol{b}=-2\hat{x}-\hat{y}+3\hat{z}$ and $\boldsymbol{c}=3\hat{x}-\hat{y}-3\hat{z}$.

Solution: We use the symmetries in (1.54) and result (1.51):

$$\begin{aligned} \boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} &= \boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ -2 & -1 & 3 \\ 3 & -1 & -3 \end{vmatrix} \\ &= 14 \end{aligned}$$

V1.10.2. Find the equation of the plane through $Q(3,-1,2)$, $S(1,-1,-3)$ and $T(4,-3,1)$.

Figure 1.44

Solution: In Figure 1.44 the flat plane is shown by which triangle $\triangle QST$ is contained; $P(x,y,z)$ is an arbitrary point in the plane. The equation of the plane is the algebraic relation between x, y and z, and we can establish this relation by noting that (for instance) vectors $\boldsymbol{p}$, $\boldsymbol{s}$ and $\boldsymbol{t}$ are coplanar. Therefore, we first write these three vectors in component form,

$$\begin{aligned}
\boldsymbol{p} &= \overline{QP} = (x-3)\hat{x} + (y+1)\hat{y} + (z-2)\hat{z}, \\
\boldsymbol{s} &= \overline{QS} = -2\hat{x} - 5\hat{z}, \\
\boldsymbol{t} &= \overline{QT} = \hat{x} - 2\hat{y} - 1\hat{z},
\end{aligned}$$

after which we demand that

$$\begin{aligned}
0 &= \boldsymbol{p} \cdot \boldsymbol{s} \times \boldsymbol{t} \\
&= \begin{vmatrix} x-3 & y+1 & z-2 \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} \\
&= 0 - 5(y+1) + 4(z-2) - 0 - 10(x-3) - 2(y+1).
\end{aligned}$$

Therefore, the equation of the plane is

$$10x + 7y - 4z = 15.$$

Problems

P1.10.1. Calculate $(2\hat{x} - 3\hat{y}) \cdot (\hat{x} + \hat{y} - \hat{z}) \times (3\hat{x} - \hat{z})$.

P1.10.2. Calculate the volume of the parallelepiped with sides $\boldsymbol{a} = 2\hat{x} - 3\hat{y} + 4\hat{z}$, $\boldsymbol{b} = \hat{x} + 2\hat{y} - \hat{z}$ and $\boldsymbol{c} = 3\hat{x} - \hat{y} + 2\hat{z}$.

P1.10.3. Determine the constant p so that vectors $2\hat{x} - \hat{y} + \hat{z}$, $\hat{x} + 2\hat{y} - 3\hat{z}$ and $3\hat{x} + p\hat{y} + 5\hat{z}$ are coplanar.

P1.10.4. Find the volume of a tetrahedron, the vertexes of the base of which are $(-2, 6, 0)$, $(3, 2, 1)$ and $(0, 0, 5)$ and the apex of which is at the origin. (Hint: volume $= \frac{1}{3}$base area $\times$ height)

P1.10.5. If $\boldsymbol{a} = \lambda_1\boldsymbol{l} + \mu_1\boldsymbol{m} + \nu_1\boldsymbol{n}$, $\boldsymbol{b} = \lambda_2\boldsymbol{l} + \mu_2\boldsymbol{m} + \nu_2\boldsymbol{n}$ and $\boldsymbol{c} = \lambda_3\boldsymbol{l} + \mu_3\boldsymbol{m} + \nu_3\boldsymbol{n}$, show that

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} (\boldsymbol{l} \cdot \boldsymbol{m} \times \boldsymbol{n}).$$

P1.10.6. Calculate the shortest distance between the origin and the plane that contains the points $(3, -2, -1)$, $(1, 3, 4)$ and $(2, 1, -2)$.

P1.10.7. Given the points $P(3, 2, 1)$, $Q(1, 1, 2)$, $R(-2, -1, -2)$ and $S(0, 1, -4)$, calculate the shortest distance between lines PQ and RS.

1.10.2 The Vector Triple Product

If we consider the vector triple product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, we ask ourselves immediately whether the order in which we calculate the two vector products is important or not, that is, whether or not we could also write the product as $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$. To be able to answer this question, we make a definite *choice*—as shown in Figure 1.45—of reference system: the Y axis is chosen in the same direction as $\boldsymbol{b}$ and the

Figure 1.45

X axis is chosen so that $\boldsymbol{c}$ lies in the XY plane. Vector $\boldsymbol{a}$'s orientation is then *arbitrary*. Since $\boldsymbol{b} \times \boldsymbol{c}$ is perpendicular to $\boldsymbol{b}$ and $\boldsymbol{c}$ by definition, it is directed in the positive Z direction. Vectors $\boldsymbol{a}$ and $\boldsymbol{b} \times \boldsymbol{c}$ therefore lie in a plane that stands perpendicular to the XY plane, and any line perpendicular to this will be *parallel* to the XY plane. But $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b} \times \boldsymbol{c}$. The vector $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ therefore is parallel to the XY plane, that is, the plane that contains $\boldsymbol{b}$ and $\boldsymbol{c}$. In the same way we can show that $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ is parallel to the plane that contains $\boldsymbol{a}$ and $\boldsymbol{b}$. Therefore, in general, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ have different directions. It follows from this that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}, \tag{1.55}$$

that is, the *vector product is not associative.*

The geometric argument stated above enables us to describe vector product $\boldsymbol{d} := \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ in a very useful way. Since $\boldsymbol{d}$ lies in the same plane as $\boldsymbol{b}$ and $\boldsymbol{c}$, it is clear from Figure 1.46 that we can construct a parellelogram with $\boldsymbol{d}$ as diagonal and with sides that are parallel to $\boldsymbol{b}$ and $\boldsymbol{c}$.

Remark. There is one special case where we cannot do this, that is, the case in which $\boldsymbol{b}$ and $\boldsymbol{c}$ are parallel or opposed. Then we have $\boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{0}$ and hence $\boldsymbol{d} = \boldsymbol{0}$.

It follows from Figure 1.46 that $\boldsymbol{d}$ can be written as a vector sum:

$$\boldsymbol{d} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \beta \boldsymbol{b} + \gamma \boldsymbol{c}. \tag{1.56}$$

Figure 1.46

On both sides of this equation we take the scalar product with $\boldsymbol{a}$:

$$\boldsymbol{a} \cdot \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \beta(\boldsymbol{a} \cdot \boldsymbol{b}) + \gamma(\boldsymbol{a} \cdot \boldsymbol{c})$$

On the left-hand side of the previous equation, we now have a scalar triple product in which the same vector ($\boldsymbol{a}$) occurs twice. We saw in §1.10.1 that such a triple product is identically equal to zero, so that

$$\beta(\boldsymbol{a} \cdot \boldsymbol{b}) + \gamma(\boldsymbol{a} \cdot \boldsymbol{c}) = 0.$$

We now have one equation in the two unknows β and γ and we can only solve for the ratio between them:

$$\frac{\beta}{\gamma} = -\frac{\boldsymbol{a} \cdot \boldsymbol{c}}{\boldsymbol{a} \cdot \boldsymbol{b}}$$

Since we have now, in essence, reduced the number of unknowns in the problem to one (the relation $\frac{\beta}{\gamma}$), it is natural to include another unknown, say λ, in such a way that the right-hand side of (1.56) has a symmetrical form. We do this by letting

$$\beta = \lambda(\boldsymbol{a} \cdot \boldsymbol{c}), \qquad \gamma = -\lambda(\boldsymbol{a} \cdot \boldsymbol{b}).$$

Then (1.56) becomes

$$\boldsymbol{d} = \lambda[(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}]. \tag{1.57}$$

We now find λ by comparing the x components of the vectors on both sides of (1.57). Let $\boldsymbol{e} \coloneqq \boldsymbol{b} \times \boldsymbol{c}$; it follows from (1.50) for the x components on the *left-hand* side of (1.57):

$$\begin{aligned} d_x &= a_y e_z - a_z e_y \\ &= a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) \\ &= a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z \end{aligned}$$

Similarly, it follows from (1.41) for the x component on the *right-hand* side of (1.57):

$$\begin{aligned} \lambda[(\boldsymbol{a}\cdot\boldsymbol{c})b_x - (\boldsymbol{a}\cdot\boldsymbol{b})c_x] &= \lambda[(a_x c_x + a_y c_y + a_z c_z)b_x - (a_x b_x + a_y b_y + a_z b_z)c_x] \\ &= \lambda[a_y b_x c_y + a_z b_x c_z - a_y b_y c_x - a_z b_z c_x] \end{aligned}$$

Since the x component on both sides of the equation must be equal, it follows that $\lambda \equiv 1$, and (1.57) yields a very important identity:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{1.58}$$

Problems

P1.10.8. Prove that $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$.

P1.10.9. Prove that $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{0}$.

P1.10.10. Prove that $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{c}\times\boldsymbol{a}) = (\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})^2$.

P1.10.11. If $\hat{e}\times\boldsymbol{b} = \boldsymbol{c}$ and $\boldsymbol{b}\times\boldsymbol{c} = \hat{e}$, prove that $\boldsymbol{c}\times\hat{e} = \boldsymbol{b}$.

P1.10.12. Solve the following equation for $\boldsymbol{r}$:

$$\begin{aligned} \boldsymbol{r}\cdot\boldsymbol{a} &= \alpha \\ \boldsymbol{r}\times\boldsymbol{b} &= \boldsymbol{c} \end{aligned}$$

where $\boldsymbol{a}\cdot\boldsymbol{b} \neq 0$.

P1.10.13. If $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{c}\times\boldsymbol{d}$ and $\boldsymbol{a}\times\boldsymbol{c} = \boldsymbol{b}\times\boldsymbol{d}$, prove that $\boldsymbol{a}-\boldsymbol{d}$ is parallel to $\boldsymbol{b}-\boldsymbol{c}$.

P1.10.14. Prove that $\boldsymbol{a}$ satisfies the following identity:

$$\boldsymbol{a} = \frac{1}{2}[\hat{x} \times (\boldsymbol{a} \times \hat{x}) + \hat{y} \times (\boldsymbol{a} \times \hat{y}) + \hat{z} \times (\boldsymbol{a} \times \hat{z})]$$

P1.10.15. Show that $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c})\boldsymbol{b}$.

P1.10.16. Simplify $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{c} + \boldsymbol{a})$.