

Alternate notation

We adopt the following notation for Cartesian vectors:

$$\begin{aligned}\langle a, b \rangle &= a\hat{x} + b\hat{y}, \\ \langle a, b, c \rangle &= a\hat{x} + b\hat{y} + c\hat{z}.\end{aligned}$$

With this notation, we find the following expressions for \hat{x} :

$$\begin{aligned}\hat{x} &= 1\hat{x} + 0\hat{y} = \langle 1, 0 \rangle, && \text{[two dimensions]} \\ \hat{x} &= 1\hat{x} + 0\hat{y} + 0\hat{z} = \langle 1, 0, 0 \rangle. && \text{[three dimensions]}\end{aligned}$$

Similarly,

$$\hat{y} = 0\hat{x} + 1\hat{y} = \langle 0, 1 \rangle$$

and

$$\begin{aligned}\hat{y} &= 0\hat{x} + 1\hat{y} + 0\hat{z} = \langle 0, 1, 0 \rangle, \\ \hat{z} &= 0\hat{x} + 0\hat{y} + 1\hat{z} = \langle 0, 0, 1 \rangle.\end{aligned}$$

Let $\mathbf{a} = \langle a_x, a_y \rangle$ and $\mathbf{b} = \langle b_x, b_y \rangle$ be any two vectors. The scalar product, in component form, is

$$\begin{aligned}\langle a_x, a_y \rangle \cdot \langle b_x, b_y \rangle &= (a_x\hat{x} + a_y\hat{y}) \cdot (b_x\hat{x} + b_y\hat{y}) \\ &= a_x\hat{x} \cdot b_x\hat{x} + a_x\hat{x} \cdot b_y\hat{y} + a_y\hat{y} \cdot a_x\hat{x} + a_y\hat{y} \cdot b_y\hat{y} \\ &= (a_xb_x)\hat{x} \cdot \hat{x} + (a_xb_y)\hat{x} \cdot \hat{y} + (a_yb_x)\hat{y} \cdot \hat{x} + (a_yb_y)\hat{y} \cdot \hat{y} \\ &= a_xb_x + a_yb_y.\end{aligned}$$

Note that

$$\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = \langle a_x, a_y \rangle \cdot \langle a_x, a_y \rangle = a_x^2 + a_y^2 = |\mathbf{a}|^2.$$

In three dimensions,

$$\begin{aligned}\langle a_x, a_y, a_z \rangle \cdot \langle b_x, b_y, b_z \rangle &= a_xb_x + a_yb_y + a_zb_z \\ \langle a_x, a_y, a_z \rangle \cdot \langle a_x, a_y, a_z \rangle &= a_x^2 + a_y^2 + a_z^2\end{aligned}$$

Observe that

$$\begin{aligned}\hat{x} \cdot \hat{y} &= \langle 1, 0 \rangle \cdot \langle 1, 0 \rangle = 1^2 + 0^2 = 1, \\ \hat{y} \cdot \hat{y} &= \langle 0, 1 \rangle \cdot \langle 0, 1 \rangle = 0^2 + 1^2 = 1, \\ \hat{x} \cdot \hat{z} &= \langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = (1)(0) + (0)(1) = 0.\end{aligned}$$

Similar results are obtained in three dimensions, for example

$$\hat{x} \cdot \hat{z} = \langle 1, 0, 0 \rangle \cdot \langle 0, 0, 1 \rangle = (1)(0) + (0)(0) + (0)(1) = 0.$$